Workbook

An Introduction to Mathematical Economics Part 2

$$\frac{\partial C^*}{\partial W} = L^*$$

$$Q_2^* = -\frac{\frac{\partial U^*}{\partial P_2}}{\frac{\partial U^*}{\partial Y}}$$

Loglinear Publishing

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Terms of Use

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Chapter 1

Implicit Functions and Total Differentials

1.1 Linear Models

Problem 1 *Consider the following implicit function:*

$$g(y, x_1, x_2) = 3y - 6x_1 - 5x_2 = 0.$$

Solve for the reduced form and find $\frac{\partial y_1}{\partial x_1}$ *and* $\frac{\partial y_1}{\partial x_2}$ *from the reduced form. Now find* $\frac{\partial y_1}{\partial x_1}$ *and* $\frac{\partial y_1}{\partial x_2}$ *from the total differential.*

Answer

We have

$$g(y, x_1, x_2) = 3y - 6x_1 - 5x_2 = 0 \Longrightarrow y = \frac{6}{3}x_1 + \frac{5}{3}x_2$$

so the reduced form is:

$$y = f(x_1, x_2) = 2x_1 + \frac{5}{3}x_2$$

so that:

$$\frac{\partial y}{\partial x_1} = 2, \ \frac{\partial y}{\partial x_2} = \frac{5}{3}.$$

Alternatively from the total differential:

$$3dy - 6dx_1 - 5dx_2 = 0$$

we have setting $dx_2 = 0$ and replacing the $d's$ with $\partial's$:

$$3\partial y - 6\partial x_1 = 0 \Longrightarrow \frac{\partial y}{\partial x_1} = 2$$

while setting $dx_1 = 0$ and replacing the $d's$ with $\partial's$:

$$3\partial y - 5\partial x_2 = 0 \Longrightarrow \frac{\partial y}{\partial x_1} = \frac{5}{3}.$$

Problem 2 *Consider the following system of equations:*

$$\begin{aligned} 3y_1 + 4y_2 - 6x_1 - 5x_2 - 5 &= 0 \\ 2y_1 + 5y_2 + 6x_1 + 5x_2 + 5 &= 0. \end{aligned}$$

Write these equations in matrix form and solve for the reduced form. Find $\frac{\partial y_1}{\partial x_1}$ *and* $\frac{\partial y_2}{\partial x_1}$ *from the reduced form. Now find* $\frac{\partial y_1}{\partial x_1}$ *and* $\frac{\partial y_2}{\partial x_1}$ *from the total differential using Cramer's rule.*

Answer

We have:

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} -6 & -5 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus:

$$\det[A] = \det \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} = 7 \neq 0$$

so that A is non-singular. We directly calculate the reduced form as:

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= -\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -6 & -5 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} \frac{54}{7}x_1 + \frac{45}{7}x_2 + \frac{45}{7} \\ -\frac{30}{7}x_1 - \frac{25}{7}x_2 - \frac{25}{7} \end{bmatrix} \end{aligned}$$

so that:

$$\begin{aligned} y_1 &= f_1(x_1, x_2) = \frac{54}{7}x_1 + \frac{45}{7}x_2 + \frac{45}{7} \\ y_2 &= f_2(x_1, x_2) = -\frac{30}{7}x_1 - \frac{25}{7}x_2 - \frac{25}{7} \end{aligned}$$

and hence:

$$\frac{\partial y_1}{\partial x_1} = \frac{54}{7}, \ \frac{\partial y_2}{\partial x_1} = -\frac{30}{7}.$$

Alternatively we have from the total differential that:

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} + \begin{bmatrix} -6 & -5 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that setting $dx_2 = 0$ and replacing $d's$ with $\partial' s$ we have:

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \partial y_1 \\ \partial y_2 \end{bmatrix} + \begin{bmatrix} -6 & -5 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or:

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \partial y_1 \\ \partial y_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} \partial x_1$$

so that dividing both sides by ∂x_1 we obtain:

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

Thus using Cramer's rule:

$$\begin{aligned} \frac{\partial y_1}{\partial x_1} &= \frac{\det \begin{bmatrix} 6 & 4 \\ -6 & 5 \end{bmatrix}}{\det \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}} = \frac{54}{7} \\ \frac{\partial y_2}{\partial x_1} &= \frac{\det \begin{bmatrix} 3 & 6 \\ 2 & -6 \end{bmatrix}}{\det \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}} = -\frac{30}{7}. \end{aligned}$$

The following 3 problems are based on the following information:

Consider the following set of implicit functions for $y_1 > 0$:

$$\begin{aligned} g_1(y_1, y_2, x_1, x_2) &= \ln(y_1) - 2y_2 - 3x_1 - 4x_2 + 6 = 0 \\ g_2(y_1, y_2, x_1, x_2) &= y_2 - \ln(y_1) - 2x_1 + 3x_2 - 5 = 0 \end{aligned}$$

with reduced form $y_1 = f_1(x_1, x_2)$ and $y_2 = f_1(x_1, x_2)$.

Problem 3 *Show how this system can be rewritten as a linear set of equations: $A\tilde{y} + Bx + c = 0$, what are $A, \tilde{y}, B, x,$ and c ? Show that A is nonsingular by calculating its determinant.*

Answer

Setting $\tilde{y}_1 = \ln(y_1)$ and $\tilde{y}_2 = y_2$ we have:

$$\begin{aligned} g_1(\tilde{y}_1, \tilde{y}_2, x_1, x_2) &= \tilde{y}_1 - 2\tilde{y}_2 - 3x_1 - 4x_2 + 6 = 0 \\ g_2(\tilde{y}_1, \tilde{y}_2, x_1, x_2) &= \tilde{y}_2 - \tilde{y}_1 - 2x_1 + 3x_2 - 5 = 0 \end{aligned}$$

so that:

$$\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus:

$$\det [A] = \det \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = -1 \neq 0$$

so that A is non-singular.

Problem 4 *Solve for the reduced form $\tilde{y} = Dx + e$ and determine*

$$\frac{\partial y_1}{\partial x_2}.$$

What does the sign of the multiplier tell you about the relationship between x_2 and y_1?

Answer

We have:

$$\begin{aligned} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} &= -\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} -7x_1 + 2x_2 - 4 \\ -5x_1 - x_2 + 1 \end{bmatrix} \end{aligned}$$

so that:

$$\begin{aligned} \tilde{y}_1 &= -7x_1 + 2x_2 - 4 \Longrightarrow y_1 = \exp(\tilde{y}_1) = e^{-7x_1 + 2x_2 - 4} \\ \tilde{y}_2 &= -5x_1 - x_2 + 1 \Longrightarrow y_2 = \tilde{y}_2 = -5x_1 - x_2 + 1 \end{aligned}$$

and

$$\frac{\partial y_1}{\partial x_2} = 2e^{-7x_1 + 2x_2 - 4} = 2y_1 > 0.$$

Therefore there exists a positive relationship between x_2 and y_1.

Problem 5 *Use the total differential on the original model solve for $\frac{\partial y_1}{\partial x_2}$ using Cramer's rule and show that $\frac{\partial y_1}{\partial x_2} > 0$.*

The total differential is:

$$\begin{aligned} \frac{1}{y_1} dy_1 - 2dy_2 - 3dx_1 - 4dx_2 &= 0 \\ -\frac{1}{y_1} dy_1 + dy_2 - 2dx_1 + 3dx_2 &= 0 \end{aligned}$$

and writing this in matrix notation we have:

$$\begin{bmatrix} \frac{1}{y_1} & -2 \\ -\frac{1}{y_1} & 1 \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that setting $dx_1 = 0$ we have:

$$\begin{bmatrix} \frac{1}{y_1} & -2 \\ -\frac{1}{y_1} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

so that:

$$\frac{\partial y_1}{\partial x_2} = \frac{\det \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{\det \begin{bmatrix} \frac{1}{y_1} & -2 \\ -\frac{1}{y_1} & 1 \end{bmatrix}} = 2y_1 > 0.$$

The following 2 problems are based on the following information:

Consider the following system of implicit linear functions:

$$\begin{aligned} 3y_1 + 2y_2 + 4y_3 - 6x_1 + 2x_2 - 4 &= 0 \\ 2y_1 - 5y_2 + 2y_3 + x_1 - x_2 + 3 &= 0 \\ y_1 + 2y_2 - y_3 + 2x_1 - 3x_2 - 2 &= 0 \end{aligned}$$

Problem 6 *If this system is written as $Ay + Bx + c = 0$, what are $A, y, B, x,$ and c? Show that A is nonsingular by calculating its determinant.*

Answer

In matrix notation we have:

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A is nonsingular since:

$$\det \begin{bmatrix} 3 & 2 & 4 \\ 2 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix} = 47 \neq 0.$$

Problem 7 *If we write the reduced form as $y = Dx + E$, calculate $D = -A^{-1}B$ and $E = -A^{-1}c$. From the matrix D determine the multipliers:*

$$\frac{\partial y_i}{\partial x_j} \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2.$$

What do the signs of each of these multipliers tell you?

Answer

We have:

$$\begin{aligned} D &= -\begin{bmatrix} 3 & 2 & 4 \\ 2 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -6 & 2 \\ 1 & -1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{52}{47} & \frac{80}{47} \\ \frac{27}{47} & -\frac{9}{47} \\ \frac{96}{47} & -\frac{79}{47} \end{bmatrix} \\ E &= -\begin{bmatrix} 3 & 2 & 4 \\ 2 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{22}{47} \\ \frac{41}{47} \\ \frac{10}{47} \end{bmatrix} \end{aligned}$$

so the reduced form is:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{52}{47} & \frac{80}{47} \\ \frac{27}{47} & -\frac{9}{47} \\ \frac{96}{47} & -\frac{79}{47} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{22}{47} \\ \frac{41}{47} \\ \frac{10}{47} \end{bmatrix}$$

or:

$$\begin{aligned} y_1 &= f_1(x_1, x_2, x_3) = -\frac{52}{47}x_1 + \frac{80}{47}x_2 + \frac{22}{47} \\ y_2 &= f_2(x_1, x_2, x_3) = \frac{27}{47}x_1 - \frac{9}{47}x_2 + \frac{41}{47} \\ y_3 &= f_3(x_1, x_2, x_3) = \frac{96}{47}x_1 - \frac{79}{47}x_2 + \frac{10}{47}. \end{aligned}$$

It follows then that:

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} \end{bmatrix} = D = \begin{bmatrix} -\frac{52}{47} & \frac{80}{47} \\ \frac{27}{47} & -\frac{9}{47} \\ \frac{96}{47} & -\frac{79}{47} \end{bmatrix}.$$

Thus for example $\frac{\partial y_1}{\partial x_2} = \frac{80}{47} > 0$ implies that there is a positive relationship between x_2 and y_1.

Problem 8 *Write out the total differential $Ady + Bdx = 0$ and solve for the multiplier $\frac{\partial y_2}{\partial x_1}$ using Cramer's rule.*

Answer

We have:

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 1 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that setting $dx_1 = 0$:

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & -5 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

so that:

$$\frac{\partial y_1}{\partial x_2} = \frac{\det\begin{bmatrix} -2 & 2 & 4 \\ 1 & -5 & 2 \\ 3 & 2 & -1 \end{bmatrix}}{47} = \frac{80}{47}.$$

$$\frac{\partial y_2}{\partial x_2} = \frac{\det\begin{bmatrix} 3 & -2 & 4 \\ 2 & 1 & 2 \\ 1 & 3 & -1 \end{bmatrix}}{47} = -\frac{9}{47}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{\det\begin{bmatrix} 3 & 2 & -2 \\ 2 & -5 & 1 \\ 1 & 2 & 3 \end{bmatrix}}{47} = -\frac{79}{47}.$$

The following 4 problems are based on the following information:

Consider the following set of linear equations:

$$\begin{aligned} 5y_1 - 2y_2 + 3y_3 + 4x_1 + 2x_2 + x_3 &= 0 \\ -5y_2 + 2y_3 + 4x_1 - x_2 &= 0 \\ 4y_3 - x_1 - x_2 - x_3 &= 0. \end{aligned}$$

Problem 9 *If this system is written as $Ay + Bx + c = 0$, what are A, y, B, x, c and 0? Show that A is nonsingular by calculating its determinant. Show that A is upper triangular?*

Answer

Here we have $c = 0$. In matrix notation we have:

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 1 \\ 4 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that A is upper triangular since the elements below the diagonal are all 0 and it is nonsingular since:

$$\det\begin{bmatrix} 5 & -2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 4 \end{bmatrix} = 5 \times (-5) \times 4 = -100 \neq 0.$$

Problem 10 *Recursive calculations are methods whereby one problem is solved using the solution to the previous problem. There is a close connect with triangular matrices and recursive calculations. Show in particular that it is possible to solve for the reduced form by first solving for y_3, then solving for y_2 using the solution for y_3, and then solving for y_1 using the solution for y_2 and y_3.*

Answer

Note that the third equation has no y_2 or y_1. We can therefore easily calculate y_3 as:

$$4y_3 - x_1 - x_2 - x_3 = 0 \Longrightarrow y_3 = \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3.$$

Now to calculate y_2 we use our solution for y_3 and the second equation, which contains no y_1, as:

$$\begin{aligned} -5y_2 + 2y_3 + 4x_1 - x_2 &= 0 \\ &\Longrightarrow 5y_2 = 2y_3 + 4x_1 - x_2 \\ &\Longrightarrow 5y_2 = 2\underbrace{\left(\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3\right)}_{y_3} + 4x_1 - x_2 \\ &\Longrightarrow y_2 = \frac{3}{10}x_1 - \frac{1}{10}x_2 + \frac{1}{10}x_3. \end{aligned}$$

Now that we have y_2 and y_3 we can calculate y_1 from the first equation as:

$$\begin{aligned} 5y_1 - 2y_2 + 3y_3 + 4x_1 + 2x_2 + x_3 &= 0 \\ &\Longrightarrow 5y_1 = 2y_2 - 3y_3 - 4x_1 - 2x_2 - x_3 \\ &\Longrightarrow 5y_1 = 2\left(\frac{3}{10}x_1 - \frac{1}{10}x_2 + \frac{1}{10}x_3\right) - 3\left(\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3\right) \\ &\quad -4x_1 - 2x_2 - x_3 \\ &\Longrightarrow y_1 = -\frac{83}{100}x_1 - \frac{59}{100}x_2 - \frac{31}{100}x_3. \end{aligned}$$

Problem 11 *If we write the reduced form as $y = Dx + E$, calculate $D = -A^{-1}B$ and $E = -A^{-1}c$. From the matrix D determine the multipliers:*

$$\frac{\partial y_i}{\partial x_j} \text{ for } i = 1, 2, 3 \text{ and } j = 1, 2.$$

What do the signs of each of these multipliers tell you?

Answer

The inverse of an upper triangular matrix is also upper triangular and hence easier to calculate. We have:

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 4 \end{bmatrix}^{-1} = \frac{-1}{100}\begin{bmatrix} -20 & 8 & 11 \\ 0 & 20 & -10 \\ 0 & 0 & -25 \end{bmatrix}.$$

We have:

$$\begin{aligned} D &= -\begin{bmatrix} 5 & -2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 2 & 1 \\ 4 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \\ &= \frac{1}{100}\begin{bmatrix} -20 & 8 & 11 \\ 0 & 20 & -10 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 4 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{59}{100} & -\frac{59}{100} & -\frac{31}{100} \\ \frac{9}{10} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}. \end{aligned}$$

Since $c = 0$ it follows that $E = -D^{-1}c = 0$. The reduced form is therefore:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{59}{100} & -\frac{59}{100} & -\frac{31}{100} \\ \frac{9}{10} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or a reduced form:

$$\begin{aligned} y_1 &= f_1(x_1, x_2, x_3) = -\frac{59}{100}x_1 - \frac{59}{100}x_2 - \frac{31}{100}x_3 \\ y_2 &= f_2(x_1, x_2, x_3) = \frac{9}{10}x_1 - \frac{1}{10}x_2 + \frac{1}{10}x_3 \\ y_3 &= f_3(x_1, x_2, x_3) = \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3. \end{aligned}$$

It follows then that:

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{bmatrix} = D = \begin{bmatrix} -\frac{59}{100} & -\frac{59}{100} & -\frac{31}{100} \\ \frac{9}{10} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Thus for example $\frac{\partial y_2}{\partial x_3} = \frac{1}{10} > 0$ implies that there is a positive relationship between x_3 and y_2.

Problem 12 *Write out the total differential $Ady + Bdx = 0$ and solve for the multipliers $\frac{\partial y_1}{\partial x_3}, \frac{\partial y_2}{\partial x_3}, \frac{\partial y_3}{\partial x_3}$ using Cramer's rule.*

Answer

We have:

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 1 \\ 4 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that setting $dx_1 = dx_2 = 0$ that:

$$\begin{bmatrix} 5 & -2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial y_1}{\partial x_3} &= \frac{\det \begin{bmatrix} -1 & -2 & 3 \\ 0 & -5 & 2 \\ 1 & 0 & 4 \end{bmatrix}}{-100} = -\frac{31}{100}. \\ \frac{\partial y_2}{\partial x_3} &= \frac{\det \begin{bmatrix} 5 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}}{-100} = \frac{1}{10} \\ \frac{\partial y_3}{\partial x_3} &= \frac{\det \begin{bmatrix} 5 & -2 & -1 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{-100} = \frac{1}{4}. \end{aligned}$$

1.2 Non-Linear Models

Problem 13 *Write the equation:* $\ln(y) = -\frac{1}{2}x^2$ *as an explicit function and as an implicit function. Calculate the total differential and from this calculate* $\frac{dy}{dx}$.

Answer

The explicitly and implicit functions are:

$$\begin{aligned} y &= f(x) = e^{-\frac{1}{2}x^2} \\ g(y,x) &= \ln(y) + \frac{1}{2}x^2 = 0. \end{aligned}$$

The direct calculation of the derivative yields:

$$\frac{dy}{dx} = -xe^{-\frac{1}{2}x^2} = -xy.$$

Using the total differential we have:

$$\frac{1}{y}dy + xdx = 0 \Longrightarrow \frac{dy}{dx} = -xy.$$

Problem 14 *Given the implicit function*

$$g(y,x) = \sqrt{y}\ln(y) - 2e^{-x} = 0$$

where $x > 0$ *and* $y > 1$, *calculate the total differential and show from this that* $\frac{dy}{dx} < 0$. *Can you find the reduced for* $y = f(x)$ *directly?*

Answer

The total differential is:

$$\left(\frac{1}{\sqrt{y}}+\frac{\ln(y)}{2\sqrt{y}}\right)dy+2e^{-x}dx=0$$

so that:

$$\frac{dy}{dx}=\frac{-2e^{-x}}{\left(\frac{1}{\sqrt{y}}+\frac{\ln(y)}{2\sqrt{y}}\right)}=\frac{-2\sqrt{y}e^{-x}}{(1+\ln(y))}<0$$

since the numerator is negative and the denominator is positive since:

$$y>1\Longrightarrow 1+\ln(y)>0.$$

It is not possible to directly calculate the reduced form since there is no way of simultaneously liberating y from $\ln(y)$ and $\sqrt{y}$ in $\sqrt{y}\ln(y)$.

The following 3 problems are based on the following information:

Consider the implicit functions:

$$\begin{aligned} g_1(y_1,y_2,x_1,x_2) &= (y_1)^{-\frac{1}{2}}(y_2)^{\frac{1}{4}}-2x_1=0 \\ g_2(y_1,y_2,x_1,x_2) &= (y_1)^{\frac{1}{2}}(y_2)^{-\frac{3}{4}}-4x_2=0 \end{aligned}$$

where y_1 and y_2 are the endogenous variables and x_1 and x_2 are the exogenous variables.

Problem 15 *Use the* $\ln(\,)$ *function to convert this into a system of linear equations and use this to find the reduced form. From the reduced form find* $\frac{\partial y_1}{\partial x_1}$ *and* $\frac{\partial y_2}{\partial x_1}$*, show that* $\frac{\partial y_1}{\partial x_1}<0$ *and that* $\frac{\partial y_2}{\partial x_1}=\frac{\partial y_1}{\partial x_2}$.

Answer

We have:

$$\begin{aligned} (y_1)^{-\frac{1}{2}}(y_2)^{\frac{1}{4}}-2x_1 &= 0 \\ &\Longrightarrow (y_1)^{-\frac{1}{2}}(y_2)^{\frac{1}{4}}=2x_1 \\ &\Longrightarrow -\frac{1}{2}\ln(y_1)+\frac{1}{4}\ln(y_2)-\ln(x_1)-\ln(2)=0 \\ (y_1)^{\frac{1}{2}}(y_2)^{-\frac{3}{4}}-4x_2 &= 0 \\ &\Longrightarrow (y_1)^{\frac{1}{2}}(y_2)^{-\frac{3}{4}}=4x_2 \\ &\Longrightarrow \frac{1}{2}\ln(y_1)-\frac{3}{4}\ln(y_2)-\ln(x_2)-\ln(4)=0 \end{aligned}$$

so that setting $\tilde{y}_1 = \ln(y_1)$, $\tilde{y}_2 = \ln(y_2)$, $\tilde{x}_1 = \ln(x_1)$, $\tilde{x}_2 = \ln(x_2)$ we have:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{3}{4} \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} -\ln(2) \\ -\ln(4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that simplifying and solving we have:

$$\begin{aligned} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{3}{4} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{3}{4} \end{bmatrix}^{-1} \begin{bmatrix} \ln(2) \\ \ln(4) \end{bmatrix} \\ &= \begin{bmatrix} -3x_1 - x_2 - 3\ln(2) - \ln(4) \\ -2x_1 - 2x_2 - 2\ln(2) - 2\ln(4) \end{bmatrix} \end{aligned}$$

or:

$$\begin{aligned} \tilde{y}_1 &= -3x_1 - x_2 - 3\ln(2) - \ln(4) \\ \tilde{y}_2 &= -2x_1 - 2x_2 - 2\ln(2) - 2\ln(4) \end{aligned}$$

so that:

$$\begin{aligned} y_1 &= f_1(x_1, x_2) = e^{\tilde{y}_1} = e^{-3x_1 - x_2 - 3\ln(2) - \ln(4)} = \frac{1}{32} e^{-3x_1 - x_2} \\ y_2 &= f_2(x_1, x_2) = e^{\tilde{y}_2} = e^{-2x_1 - 2x_2 - 2\ln(2) - 2\ln(4)} = \frac{1}{64} e^{-2x_1 - 2x_2}. \end{aligned}$$

From this you can show that:

$$\begin{aligned} \frac{\partial y_1}{\partial x_1} &= -\frac{3}{32} e^{-3x_1 - x_2} < 0, \frac{\partial y_2}{\partial x_1} = -\frac{1}{32} e^{-3x_1 - x_2} \\ \frac{\partial y_2}{\partial x_1} &= \frac{\partial y_1}{\partial x_2} = -\frac{1}{32} e^{-3x_1 - x_2}. \end{aligned}$$

Problem 16 *Calculate the total differential for this system of equations. Show that the conditions for the explicit functions to exist are satisfied.*

Answer

The total differential is:

$$\begin{aligned} \left(-\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}}\right) dy_1 + \left(\frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}}\right) dy_2 - 2dx_1 &= 0 \\ \left(\frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}}\right) dy_1 + \left(-\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}}\right) dy_2 - 4dx_2 &= 0. \end{aligned}$$

Since:

$$A = \begin{bmatrix} -\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}} & \frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} \\ \frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} & -\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}} \end{bmatrix}$$

and

$$\det\begin{bmatrix} -\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}} & \frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} \\ \frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} & -\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}} \end{bmatrix} = \frac{3}{8}y_1^{-1}y_2^{-\frac{6}{4}} - \frac{1}{8}y_1^{-1}y_2^{-\frac{6}{4}} = \frac{2}{8}y_1^{-1}y_2^{-\frac{6}{4}} > 0$$

by the implicit function theorem the explicit functions $y_1 = f_1(x_1, x_2)$ and $y_2 = f_2(x_1, x_2)$ exist (which we already know since we calculated them in the previous problem!).

Problem 17 *Use the total differential to calculate* $\frac{\partial y_1}{\partial x_1}$ *and* $\frac{\partial y_2}{\partial x_1}$ *and show that* $\frac{\partial y_1}{\partial x_1} < 0$ *and that* $\frac{\partial y_2}{\partial x_1} = \frac{\partial y_1}{\partial x_2}$.

Answer

From the total differential setting $dx_2 = 0$ we have:

$$\begin{bmatrix} -\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}} & \frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} \\ \frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} & -\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}} \end{bmatrix}\begin{bmatrix} \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

so that using Cramer's rule:

$$\frac{\partial y_1}{\partial x_1} = \frac{\det\begin{bmatrix} 2 & \frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} \\ 0 & -\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}} \end{bmatrix}}{\underbrace{\det[A]}_{+}} = \frac{-\frac{6}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}}}{\underbrace{\det[A]}_{+}} < 0$$

$$\frac{\partial y_2}{\partial x_1} = \frac{\det\begin{bmatrix} -\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}} & 2 \\ \frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} & 0 \end{bmatrix}}{\underbrace{\det[A]}_{+}} = \frac{-(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}}}{\underbrace{\det[A]}_{+}} < 0.$$

Similarly from the total differential setting $dx_1 = 0$ we have:

$$\begin{bmatrix} -\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}} & \frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} \\ \frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} & -\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}} \end{bmatrix}\begin{bmatrix} \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

so that using Cramer's rule:

$$\frac{\partial y_1}{\partial x_2} = \frac{\det\begin{bmatrix} 0 & \frac{1}{4}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} \\ 4 & -\frac{3}{4}(y_1)^{\frac{1}{2}}(y_2)^{-\frac{7}{4}} \end{bmatrix}}{\underbrace{\det[A]}_{+}} = \frac{(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}}}{\underbrace{\det[A]}_{+}} > 0$$

$$\frac{\partial y_2}{\partial x_2} = \frac{\det\begin{bmatrix} -\frac{1}{2}(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}} & 0 \\ \frac{1}{2}(y_1)^{-\frac{1}{2}}(y_2)^{-\frac{3}{4}} & 4 \end{bmatrix}}{\underbrace{\det[A]}_{+}} = \frac{-2(y_1)^{-\frac{3}{2}}(y_2)^{\frac{1}{4}}}{\underbrace{\det[A]}_{+}} < 0.$$

The following 3 problems are based on the following information:

Consider the following system of implicit functions:

$$\begin{aligned} g_1(y_1, y_2, y_3, x_1, x_2, x_3) &= 2\ln(x_3) - \ln(y_2) - \ln(y_3) = 0 \\ g_2(y_1, y_2, y_3, x_1, x_2, x_3) &= 2x_1 - \frac{y_1}{y_2} = 0 \\ g_3(y_1, y_2, y_3, x_1, x_2, x_3) &= 2x_2 - \frac{y_1}{y_3} = 0 \end{aligned}$$

where y_1, y_2 and y_3 are the endogenous variables (assumed to be positive) and x_1, x_2 and x_3 are the exogenous variables (also assumed to be positive).

Problem 18 *Calculate the total differential and write it in matrix notation. Are the conditions of the implicit function theorem which guarantee the existence of the explicit functions satisfied?*

Answer

The total differential is:

$$\begin{aligned} -\frac{1}{y_2}dy_2 - \frac{1}{y_3}dy_3 + \frac{2}{x_3}dx_3 &= 0 \\ -\frac{1}{y_2}dy_1 + \frac{y_1}{y_2^2}dy_2 + 2dx_1 &= 0 \\ -\frac{1}{y_3}dy_1 + \frac{y_1}{y_3^2}dy_3 + 2dx_2 &= 0 \end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} 0 & -\frac{1}{y_2} & -\frac{1}{y_3} \\ -\frac{1}{y_2} & \frac{y_1}{y_2^2} & 0 \\ -\frac{1}{y_3} & 0 & \frac{y_1}{y_3^2} \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{2}{x_3} \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus:

$$\det[A] = \det \begin{bmatrix} 0 & -\frac{1}{y_2} & -\frac{1}{y_3} \\ -\frac{1}{y_2} & \frac{y_1}{y_2^2} & 0 \\ -\frac{1}{y_3} & 0 & \frac{y_1}{y_3^2} \end{bmatrix} = -\frac{2}{y_2^2}\frac{y_1}{y_3^2} < 0.$$

Since $\det[A] \neq 0$ the reduced form:

$$y_1 = f_1(x_1, x_2, x_3), \; y_2 = f_2(x_1, x_2, x_3), \; y_3 = f_3(x_1, x_2, x_3)$$

exists.

Problem 19 *From the total differential calculate* $\frac{\partial y_1}{\partial x_3}$ *and* $\frac{\partial y_2}{\partial x_2}$ *and determine their signs if possible.*

Answer

Setting $dx_1 = dx_2 = 0$ we have

$$\begin{bmatrix} 0 & -\frac{1}{y_2} & -\frac{1}{y_3} \\ -\frac{1}{y_2} & \frac{y_1}{y_2^2} & 0 \\ -\frac{1}{y_3} & 0 & \frac{y_1}{y_3^2} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{x_3} \\ 0 \\ 0 \end{bmatrix}$$

so that from Cramer's rule:

$$\frac{\partial y_1}{\partial x_3} = \frac{\begin{bmatrix} -\frac{2}{x_3} & -\frac{1}{y_2} & -\frac{1}{y_3} \\ 0 & \frac{y_1}{y_2^2} & 0 \\ 0 & 0 & \frac{y_1}{y_3^2} \end{bmatrix}}{\det[A]} = \frac{-\frac{2}{x_3} \times \frac{y_1}{y_2^2} \times \frac{y_1}{y_3^2}}{\det[A]} > 0.$$

Setting $dx_1 = dx_3 = 0$ we have

$$\begin{bmatrix} 0 & -\frac{1}{y_2} & -\frac{1}{y_3} \\ -\frac{1}{y_2} & \frac{y_1}{y_2^2} & 0 \\ -\frac{1}{y_3} & 0 & \frac{y_1}{y_3^2} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

so that from Cramer's rule:

$$\frac{\partial y_2}{\partial x_2} = \frac{\det \begin{bmatrix} 0 & 0 & -\frac{1}{y_3} \\ -\frac{1}{y_2} & 0 & 0 \\ -\frac{1}{y_3} & 2 & \frac{y_1}{y_3^2} \end{bmatrix}}{\det[A]} = \frac{\frac{2}{y_2 y_3}}{\det[A]} < 0.$$

Problem 20 *Solve the three implicit functions above for the explicit functions* $y_1 = f_1(x_1, x_2, x_3)$, $y_2 = f_1(x_1, x_2, x_3)$ *and* $y_3 = f_3(x_1, x_2, x_3)$.

Answer

From the 3 implicit functions we have:

$$\begin{aligned} 2\ln(x_3) - \ln(y_2) - \ln(y_3) &= 0 \Longrightarrow y_2 y_3 = x_3^2 \\ 2x_1 - \frac{y_1}{y_2} &= 0 \Longrightarrow y_1 = 2x_1 y_2 \\ 2x_2 - \frac{y_1}{y_3} &= 0 \Longrightarrow y_1 = 2x_2 y_3 \end{aligned}$$

so that dividing the second and third results we have:

$$\frac{x_1 y_2}{x_2 y_3} = 1 \Longrightarrow y_2 = \frac{x_2}{x_1} y_3$$

and placing this in the first result:

$$\begin{aligned} y_2 y_3 &= x_3^2 \text{ and } y_2 = \frac{x_2}{x_1} y_3 \Longrightarrow \frac{x_2}{x_1} y_3^2 = x_3^2 \\ &\Longrightarrow y_3 = f_3(x_1, x_2, x_3) = x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} x_3 \end{aligned}$$

and so:

$$\begin{aligned} y_2 &= \frac{x_2}{x_1} y_3 \text{ and } y_3 = x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} x_3 \Longrightarrow y_2 = \frac{x_2}{x_1} x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} x_3 \\ &\Longrightarrow y_2 = f_2(x_1, x_2, x_3) = x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} x_3. \end{aligned}$$

Finally:

$$\begin{aligned} y_1 &= 2x_2 y_3 \text{ and } y_3 = x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} x_3 \Longrightarrow y_1 = 2x_2 x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} x_3 \\ &\Longrightarrow y_1 = f_1(x_1, x_2, x_3) = 2x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} x_3. \end{aligned}$$

Problem 21 *Consider the following system of implicit functions:*

$$\begin{aligned} g_1(y_1, y_2, x_1, x_2, x_3) &= y_1^{2x_1} - 3y_2^{x_2} + x_3 = 0 \\ g_2(y_1, y_2, x_1, x_2, x_3) &= x_1 \ln(y_1) + x_2 \ln(y_2) = 0 \end{aligned}$$

where all variables (i.e. all $y's$ and $x's$) are assumed to be positive. Calculate the total differential of this system and write it in matrix form. Show that the conditions for the implicit function theorem will be satisfied. Solve for $\frac{\partial y_1}{\partial x_3}$ using Cramer's rule and determine its sign if possible.

Answer

We have the total differential:

$$\begin{aligned} 2x_1 y_1^{2x_1-1} dy_1 - 3x_2 y_2^{x_2-1} dy_2 + 2\ln(y_1) y_1^{2x_1} dx_1 - 3\ln(y_2) y_2^{x_2} dx_2 + dx_3 &= 0 \\ \frac{x_1}{y_1} dy_1 + \frac{x_2}{y_2} dy_2 + \ln(y_1) dx_1 + \ln(y_2) dx_2 &= 0. \end{aligned}$$

In matrix form we have:

$$\begin{bmatrix} 2x_1 y_1^{2x_1-1} & -3x_2 y_2^{x_2-1} \\ \frac{x_1}{y_1} & \frac{x_2}{y_2} \end{bmatrix} \begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} + \begin{bmatrix} 2\ln(y_1) y_1^{2x_1} & -3\ln(y_2) y_2^{x_2} & 1 \\ \ln(y_1) & \ln(y_2) & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus

$$\det \begin{bmatrix} 2x_1 y_1^{2x_1-1} & -3x_2 y_2^{x_2-1} \\ \frac{x_1}{y_1} & \frac{x_2}{y_2} \end{bmatrix} = \frac{x_1 x_2 \left(2y_1^{2x_1-1} y_1 + 3y_2^{x_2-1} y_2\right)}{y_2 y_1} > 0$$

since all variables (i.e. all $y's$ and $x's$) are assumed to be positive. Since $\det[A] \neq 0$ the reduced form:

$$y_1 = f_1(x_1, x_2, x_3), \; y_2 = f_2(x_1, x_2, x_3)$$

exists.

Setting $dx_1 = dx_2 = 0$ we have:

$$\begin{bmatrix} 2x_1 y_1^{2x_1-1} & -3x_2 y_2^{x_2-1} \\ \frac{x_1}{y_1} & \frac{x_2}{y_2} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial y_1}{\partial x_3} &= \frac{\det \begin{bmatrix} 1 & -3x_2 y_2^{x_2-1} \\ 0 & \frac{x_2}{y_2} \end{bmatrix}}{\det \begin{bmatrix} 2x_1 y_1^{2x_1-1} & -3x_2 y_2^{x_2-1} \\ \frac{x_1}{y_1} & \frac{x_2}{y_2} \end{bmatrix}} = \frac{\frac{x_2}{y_2}}{x_1 x_2 \frac{2y_1^{2x_1-1} y_1 + 3y_2^{x_2-1} y_2}{y_2 y_1}} \\ &= \frac{y_1}{x_1 \left(2y_1^{2x_1} + 3y_2^{x_2}\right)} > 0. \end{aligned}$$

1.3 Supply and Demand

The following 2 problems are based on the following information:

Consider a demand and supply model where households pay a tax T for every unit they buy. Demand and supply are therefore given by:

$$\begin{aligned} Q^d &= D(P+T), \\ Q^s &= S(P). \end{aligned}$$

Letting $Q^s = Q^d = Q$ we then have:

$$\begin{aligned} Q &= D(P+T) \\ Q &= S(P) \end{aligned}$$

and

$$D_P \equiv D'(P+T) < 0, \; S_P \equiv S'(P) < 0$$

Problem 22 *Write the structural model as a system of implicit functions indicating what are the endogenous and exogenous variables. Calculate the total differential.*

Answer

We have:

$$\begin{aligned} g_1(Q,P,T) &= Q - D(P+T) = 0 \\ g_2(Q,P,T) &= Q - S(P) = 0. \end{aligned}$$

The total differential then is:

$$\begin{aligned} dQ - D'(P+T)\,dP - D'(P+T)\,dT &= 0 \\ dQ - S'(P)\,dP &= 0 \end{aligned}$$

or:

$$\begin{aligned} dQ - D_P dP - D_P dT &= 0 \\ dQ - S_P dP &= 0. \end{aligned}$$

Problem 23 *From the total differential calculate* $\frac{\partial Q}{\partial T}$ *and* $\frac{\partial P}{\partial T}$ *and show that both are negative. If* $P_C = P + T$ *is the net price that households pay, show that* $\frac{\partial P_C}{\partial T} > 0$.

Answer

We have:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \\ \det[A] &= -S_P + D_P < 0 \end{aligned}$$

and

$$\begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial T} \\ \frac{\partial P}{\partial T} \end{bmatrix} = \begin{bmatrix} D_P \\ 0 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial Q}{\partial T} &= \frac{\det \begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix}}{\det[A]} = \frac{-\overbrace{S_P}^{+}\overbrace{D_P}^{-}}{\underbrace{\det[A]}_{-}} < 0 \\ \frac{\partial P}{\partial T} &= \frac{\det \begin{bmatrix} 1 & D_P \\ 1 & 0 \end{bmatrix}}{\det[A]} = \frac{-\overbrace{D_P}^{-}}{\underbrace{\det[A]}_{-}} < 0. \end{aligned}$$

Now since $P_c = P + T$ we have:

$$\begin{aligned} \frac{\partial P_c}{\partial T} &= \frac{\partial P}{\partial T} + 1 = \frac{-D_P}{-S_P + D_P} + 1 \\ &= \frac{-S_P}{-S_P + D_P} > 0. \end{aligned}$$

The following 3 problems are based on the following information:

Suppose that demand is given by: $Q^d = D(P)$ where $D'(P) < 0$, supply is given by: $Q^s = S(P)$ where $S'(P) > 0$, and that in equilibrium $Q^d = Q^s = Q$. The government decides to impose a tax on this good such that if the household purchases one dollar of the good it must pay t dollars to the government where

$t > 0$. This means that the effective price the household must pay is $P \cdot (1+t)$. Hence in equilibrium the following two implicit functions will hold

$$\begin{aligned} Q - D(P \cdot (1+t)) &= 0 \\ Q - S(P) &= 0. \end{aligned}$$

These two implicit functions then determine the explicit functions $P(t)$ and $Q(t)$, that is, the price and quantity as a function of the tax t. To simplify the notation define:

$$D_P \equiv D'\left(P \cdot (1+t)\right) < 0, \; S_P \equiv S'\left(P\right) < 0$$

Problem 24 *Take the total differential of the two implicit functions and show that the conditions of the implicit function theorem are satisfied; that is, the appropriate matrix is nonsingular.*

Answer

The total differential then is:

$$\begin{aligned} dQ - (1+t)D'\left(P \cdot (1+t)\right) dP - D'\left(P \times (1+t)\right) \cdot P dt &= 0 \\ dQ - S'\left(P\right) dP &= 0 \end{aligned}$$

or:

$$\begin{aligned} dQ - (1+t)D_P dP - D_P P dt &= 0 \\ dQ - S_P dP &= 0 \end{aligned}$$

We have:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -(1+t)D_P \\ 1 & -S_P \end{bmatrix} \\ \det[A] &= -S_P + (1+t)D_P < 0. \end{aligned}$$

Problem 25 *Calculate* $\frac{\partial P}{\partial t}$ *and* $\frac{\partial Q}{\partial t}$ *from the last problem and determine their signs. What do these signs tell you about the impact of the tax on the market?*

Answer

We have

$$\begin{bmatrix} 1 & -(1+t)D_P \\ 1 & -S_P \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial t} \\ \frac{\partial P}{\partial t} \end{bmatrix} = \begin{bmatrix} D_P \cdot P \\ 0 \end{bmatrix}$$

so that:

$$\frac{\partial Q}{\partial t} = \frac{\det \begin{bmatrix} D_P \cdot P & -(1+t)D_P \\ 0 & -S_P \end{bmatrix}}{\det[A]} = \frac{-\overbrace{S_P}^{+}\overbrace{D_P \cdot P}^{-}}{\underbrace{\det[A]}_{-}} < 0$$

$$\frac{\partial P}{\partial t} = \frac{\det \begin{bmatrix} 1 & D_P \cdot P \\ 1 & 0 \end{bmatrix}}{\det[A]} = \frac{\overbrace{-D_P \cdot P}^{-}}{\underbrace{\det[A]}_{-}} < 0.$$

Problem 26 *The price that households pay is $P_c = P \cdot (1+t)$. Show that an increase in the tax increases P_c; that is $\frac{\partial P_c}{\partial t} > 0$.*

Answer

Now since $P_c = P \cdot (1+t)$ we have:

$$\begin{aligned} \frac{\partial P_c}{\partial t} &= \frac{\partial P}{\partial t} \cdot (1+t) + P = \frac{-D_P \cdot P}{-S_P + D_P \cdot (1+t)} \cdot (1+t) + 1 \\ &= \frac{-S_P}{-S_P + D_P \cdot (1+t)} > 0. \end{aligned}$$

The following 3 problems are based on the following information:

Consider a demand and supply model where for each unit they buy, consumers must pay a tax of T_1 and producers a tax of T_2. This results in a demand and supply model of the form:

$$\begin{aligned} Q &= D(P+T_1), \text{ 'demand curve'} \\ Q &= S(P-T_2), \text{ 'supply curve'}. \end{aligned}$$

As is usual assume that:

$$D_P \equiv D'(P+T_1) < 0,\ S_P \equiv S'(P-T_2) < 0.$$

Problem 27 *Determine the sign of $\frac{\partial P}{\partial T_1}$ and show that an increase in T_1 increases the net price: $P+T_1$ that households pay. Determine the sign of $\frac{\partial P}{\partial T_2}$ and show that an increase in T_2 decreases the net price: $P-T_2$ that firm's receive. Given that the tax revenue that the government collects is: $R = (T_1+T_2)Q$, show that the government will be indifferent between increasing T_1 and T_2 in that:*

$$\frac{\partial R}{\partial T_1} = \frac{\partial R}{\partial T_2}.$$

Answer

We have:

$$\begin{aligned} g_1(Q,P,T_1,T_2) &= Q - D(P+T_1) = 0 \\ g_2(Q,P,T_1,T_2) &= Q - S(P-T_2) = 0. \end{aligned}$$

The total differential then is:

$$\begin{aligned} dQ - D_P dP - D_P dT_1 &= 0 \\ dQ - S_P dP + S_P dT_2 &= 0. \end{aligned}$$

We have:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \\ \det[A] &= -S_P + D_P < 0 \end{aligned}$$

and setting $dT_2 = 0$

$$\frac{\partial Q}{\partial T_1} = \frac{\det \begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix}}{\det[A]} = \frac{-\overbrace{S_P}^{+}\overbrace{D_P}^{-}}{\underbrace{\det[A]}_{-}} < 0$$

$$\frac{\partial P}{\partial T_1} = \frac{\det \begin{bmatrix} 1 & D_P \\ 1 & 0 \end{bmatrix}}{\det[A]} = \frac{-\overbrace{D_P}^{-}}{\underbrace{\det[A]}_{-}} < 0.$$

Now since $P_c = P + T_1$ we have:

$$\begin{aligned} \frac{\partial P_c}{\partial T_1} &= \frac{\partial P}{\partial T_1} + 1 = \frac{-D_P}{-S_P + D_P} + 1 \\ &= \frac{-S_P}{-S_P + D_P} > 0. \end{aligned}$$

Furthermore setting $dT_1 = 0$:

$$\begin{bmatrix} 1 & -D_P \\ 1 & -S_P \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial T_2} \\ \frac{\partial P}{\partial T_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -S_P \end{bmatrix}$$

so that:

$$\frac{\partial Q}{\partial T_2} = \frac{\det \begin{bmatrix} 0 & -D_P \\ -S_P & -S_P \end{bmatrix}}{\det[A]} = \frac{-\overbrace{S_P}^{+}\overbrace{D_P}^{-}}{\underbrace{\det[A]}_{-}} < 0$$

$$\frac{\partial P}{\partial T_2} = \frac{\det \begin{bmatrix} 1 & 0 \\ 1 & -S_P \end{bmatrix}}{\det[A]} = \frac{-\overbrace{S_P}^{+}}{\underbrace{\det[A]}_{-}} > 0.$$

Now if $R = (T_1 + T_2) Q$ then :

$$\frac{\partial R}{\partial T_1} = Q + (T_1 + T_2) \frac{\partial Q}{\partial T_1} = \frac{\partial R}{\partial T_2} = Q + (T_1 + T_2) \frac{\partial Q}{\partial T_2}$$

since $\frac{\partial Q}{\partial T_1} = \frac{\partial Q}{\partial T_2}$ by the above calculation.

Problem 28 *Normally one thinks that households would prefer governments to raise taxes on firms by using T_2 while firm's would prefer that governments tax households by using T_1. Prove that the if $T = T_1 + T_2$ then households, firms and government will be indifferent to the composition of T that is how much is of T is T_1 and how much of T is T_2.*

Answer

The following 3 problems are based on the following information:

Consider a demand and supply model where for each unit they buy, consumers must pay tax at a rate of t_1 and producers a tax at a rate of t_2. This results in a demand and supply model of the form:

$$\begin{aligned} Q &= D(P \cdot (1 + t_1)), \text{ 'demand curve'} \\ Q &= S(P \cdot (1 - t_2)), \text{ 'supply curve'.} \end{aligned}$$

As is usual assume that:

$$D_P \equiv D'(P \cdot (1 + t_1)) < 0,\ S_P \equiv S'(P \cdot (1 - t_2)) < 0.$$

Problem 29 *Determine the sign of $\frac{\partial P}{\partial t_1}, \frac{\partial Q}{\partial t_1}, \frac{\partial P}{\partial t_2}$ and $\frac{\partial Q}{\partial t_2}$. The tax revenue that the government collects is: $R = (t_1 + t_2) PQ$. Find $\frac{\partial R}{\partial t_1}$ and $\frac{\partial R}{\partial t_2}$.*

Answer

We have:

$$\begin{aligned} g_1(Q, P, t_1, t_2) &= Q - D(P \cdot (1 + t_1)) = 0 \\ g_2(Q, P, t_1, t_2) &= Q - S(P \cdot (1 - t_2)) = 0. \end{aligned}$$

The total differential then is:

$$\begin{aligned} dQ - D_P \cdot (1 + t_1) dP - D_P \cdot P dt_1 &= 0 \\ dQ - S_P \cdot (1 - t_2) dP + S_P \cdot P dt_2 &= 0. \end{aligned}$$

We have:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -D_P \cdot (1 + t_1) \\ 1 & -S_P \cdot (1 - t_2) \end{bmatrix} \\ \det[A] &= -S_P \cdot (1 - t_2) + D_P \cdot (1 + t_1) < 0 \end{aligned}$$

and setting $dt_2 = 0$

$$\begin{bmatrix} 1 & -D_P \cdot (1+t_1) \\ 1 & -S_P \cdot (1-t_2) \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial t_1} \\ \frac{\partial P}{\partial t_1} \end{bmatrix} = \begin{bmatrix} D_P \cdot P \\ 0 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial Q}{\partial t_1} &= \frac{\det \begin{bmatrix} D_P \cdot P & -D_P \cdot (1+t_1) \\ 0 & -S_P \cdot (1-t_2) \end{bmatrix}}{\det [A]} = \frac{-\overbrace{D_P \cdot P}^{-}\overbrace{S_P \cdot (1-t_2)}^{+}}{\underbrace{\det [A]}_{-}} < 0 \\ \frac{\partial P}{\partial t_1} &= \frac{\det \begin{bmatrix} 1 & D_P \cdot P \\ 1 & 0 \end{bmatrix}}{\det [A]} = \frac{-\overbrace{D_P \cdot P}^{-}}{\underbrace{\det [A]}_{-}} < 0. \end{aligned}$$

Furthermore setting $dt_1 = 0$:

$$\begin{bmatrix} 1 & -D_P \cdot (1+t_1) \\ 1 & -S_P \cdot (1-t_2) \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial T_2} \\ \frac{\partial P}{\partial T_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -S_P \cdot P \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial Q}{\partial t_2} &= \frac{\det \begin{bmatrix} 0 & -D_P \cdot (1+t_1) \\ -S_P \cdot P & -S_P \cdot (1-t_2) \end{bmatrix}}{\det [A]} = \frac{-\overbrace{S_P \cdot P}^{+}\overbrace{D_P \cdot (1+t_1)}^{-}}{\underbrace{\det [A]}_{-}} < 0 \\ \frac{\partial P}{\partial t_2} &= \frac{\det \begin{bmatrix} 1 & 0 \\ 1 & -S_P \cdot P \end{bmatrix}}{\det [A]} = \frac{-\overbrace{S_P \cdot P}^{+}}{\underbrace{\det [A]}_{-}} > 0. \end{aligned}$$

Now if $R = (t_1 + t_2) PQ$ then we have:

$$\begin{aligned} \frac{\partial R}{\partial t_1} &= PQ + (t_1 + t_2) \left(\frac{\partial Q}{\partial t_1} P + \frac{\partial P}{\partial t_1} Q \right) \\ &= PQ + (t_1 + t_2) \left(\frac{-D_P \cdot P^2 S_P \cdot (1-t_2) - D_P \cdot PQ}{-S_P \cdot (1-t_2) + D_P \cdot (1+t_1)} \right) \\ \frac{\partial R}{\partial t_2} &= PQ + (t_1 + t_2) \left(\frac{\partial Q}{\partial t_2} P + \frac{\partial P}{\partial t_2} Q \right) \\ &= PQ + (t_1 + t_2) \left(\frac{-S_P \cdot P^2 D_P \cdot (1+t_1) - S_P \cdot PQ}{\det [A]} \right). \end{aligned}$$

1.4 The IS/LM Model

The following 4 problems are based on the following information:

Consider the following IS/LM model:

$$
\begin{aligned}
c^d &= c(y-t)\text{ , where } c_y \equiv c'(y-t) \text{ and } 0 < c_y < 1 \\
i^d &= i(r),\ \text{where } i_r \equiv i'(r) < 0 \\
g^d &= g_o (\text{government expenditure is exogenous}) \\
t &= t(y),\ \text{where } t_y \equiv t'(y) \text{ with } 0 < t_y < 1 \\
y &= y^d \equiv c^d + i^d + g^d \\
m^d &= l(y,r),\ \text{where } l_y \equiv \frac{\partial l(y,r)}{\partial y} > 0 \text{ and } l_r \equiv \frac{\partial l(y,r)}{\partial r} < 0 \\
m^s &= m_o^s \text{ (money supply is exogenous).} \\
m^s &= m^d
\end{aligned}
$$

Problem 30 *Show that for this model the IS curve is downward sloping and the LM curve is upward sloping.*

Answer

The IS curve is defined by:

$$y = y^d \equiv c^d + i^d + g^d = c(y - t(y)) + i(r) + g_o$$

leading to the implicit function:

$$g_1 (y, r, g_o, m_o^s) = y - c(y - t(y)) - i(r) - g_o = 0$$

with total differential:

$$(1 - c_y (1 - t_y))\, dy - i_r dr - dg_o = 0.$$

Setting $dg_0 = 0$ and solving for the slope of the IS curve: $\frac{dr}{dy}$ we obtain:

$$\frac{dr}{dy} = \frac{(1 - c_y (1 - t_y))}{i_r} < 0$$

since $i_r < 0$ and:

$$0 < 1 - c_y (1 - t_y) < 1.$$

The LM curve is defined by:

$$m_0^s = m^d = l(y, r)$$

leading to the implicit function:

$$g_2(y, r, g_o, m_o^s) = l(y, r) - m_0^s = 0$$

with total differential:

$$l_y dy + l_r dr - dm_0^s = 0.$$

Setting $dm_o^s = 0$ and solving for the slope of the LM curve: $\frac{dr}{dy}$ we obtain:

$$\frac{dr}{dy} = -\frac{l_y}{l_r} > 0$$

since $l_y > 0$ and $l_r < 0$.

Problem 31 *Using the results from the last problem and Cramer's rule, calculate* $\frac{\partial y}{\partial g_o}$ *and show that* $\frac{\partial y}{\partial g_o} > 0$. *Do the same thing with* $\frac{\partial r}{\partial M_o^s}$ *and show that* $\frac{\partial r}{\partial m_o^s} < 0$.

Answer

Setting $dm_0^s = 0$ in the total differential and writing this in matrix notation we obtain:

$$\begin{bmatrix} 1 - c_y(1 - t_y) & -i_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial g_o} \\ \frac{\partial r}{\partial g_o} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial y}{\partial g_o} &= \frac{\det \begin{bmatrix} 1 & -i'(r) \\ 0 & l_r \end{bmatrix}}{\det \begin{bmatrix} 1 - c_y(1 - t_y) & -i_r \\ l_y & l_r \end{bmatrix}} \\ &= \frac{l_r}{(1 - c_y(1 - t_y)) l_r + i_r l_y} \\ &= \frac{1}{(1 - c_y(1 - t_y)) + i_r \frac{l_y}{l_r}}. \end{aligned}$$

Since both terms in the denominator are positive; that is:

$$\begin{aligned} 1 - c_y(1 - t_y) &> 0 \\ i_r \frac{l_y}{l_r} &> 0 \end{aligned}$$

it follows that: $\frac{\partial y}{\partial g_o} > 0$.

Setting $dg_0 = 0$ in the total differential and writing this in matrix notation we obtain:

$$\begin{bmatrix} 1 - c_y(1 - t_y) & -i_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial m_o^s} \\ \frac{\partial r}{\partial m_o^s} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial r}{\partial m_o^s} &= \frac{\det \begin{bmatrix} 1 - c_y(1 - t_y) & 0 \\ l_y & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 - c_y(1 - t_y) & -i_r \\ l_y & l_r \end{bmatrix}} \\ &= \frac{1 - c_y(1 - t_y)}{(1 - c_y(1 - t_y))\, l_r + i_r l_y}. \end{aligned}$$

Since the numerator is positive while the denominator is negative, it follows that: $\frac{\partial r}{\partial m_o^s} < 0$.

Problem 32 *Show that increasing the money supply increases investment.*

Answer

We have:

$$i = i(r) \Longrightarrow \frac{\partial i}{\partial m_o^s} = i'(r) \frac{\partial r}{\partial m_o^s} = i_r \frac{\partial r}{\partial m_o^s} > 0$$

since $i_r < 0$ and from the above problem: $\frac{\partial r}{\partial m_o^s} < 0$.

Problem 33 *The government's deficit is given by:*

$$d = g_0 - t(y).$$

Now we know that an increase in government expenditure increases y and hence it will increases taxes: $t(y)$. Could increasing government expenditure then increase tax revenue so much that it would actually reduce the deficit?

Answer

We have:

$$\frac{\partial d}{\partial g_o} = \frac{\partial}{\partial g_o}(g_0 - t(y)) = 1 - t_y \frac{\partial y}{\partial g_o}.$$

We have calculated $\frac{\partial y}{\partial g_o} > 0$ above so it appears the result is ambiguous since $t_y \frac{\partial y}{\partial g_o} > 0$. However:

$$
\begin{aligned}
\frac{\partial d}{\partial g_o} &= 1 - t_y \frac{\partial y}{\partial g_o} \\
&= 1 - t_y \frac{1}{(1 - c_y (1 - t_y)) + i_r \frac{l_y}{l_r}} \\
&= \frac{(1 - c_y (1 - t_y)) + i_r \frac{l_y}{l_r} - t_y}{(1 - c_y (1 - t_y)) + i_r \frac{l_y}{l_r}} \\
&= \frac{((1 - c_y) + c_y t_y) + i_r \frac{l_y}{l_r} - t_y}{(1 - c_y (1 - t_y)) + i_r \frac{l_y}{l_r}} \\
&= \frac{(1 - c_y) + i_r \frac{l_y}{l_r} - (1 - c_y) t_y}{(1 - c_y (1 - t_y)) + i_r \frac{l_y}{l_r}} \\
&= \frac{\overbrace{(1 - c_y)}^{+}\overbrace{(1 - t_y)}^{+} + i_r \overbrace{\frac{l_y}{l_r}}^{+}}{(1 - c_y (1 - t_y)) + i_r \frac{l_y}{l_r}} > 0
\end{aligned}
$$

and so increasing government expenditure does increase the deficit.

The following 3 problems are based on the following information:
Consider the following IS/LM model:

$$
\begin{aligned}
c^d &= c(y, r), \text{ where } 0 < c_y \equiv \frac{\partial c(y, r)}{\partial y} < 1 \text{ and } c_r \equiv \frac{\partial c(y, r)}{\partial r} < 0 \\
i^d &= i_o \text{ (i.e. investment is exogenous)} \\
g^d &= g_o \text{ (i.e. government expenditure is exogenous)} \\
y &= c^d + i^d + g^d
\end{aligned}
$$

$$
\begin{aligned}
m^d &= l(y, r), \text{ where } l_y \equiv \frac{\partial l(y, r)}{\partial y} > 0 \text{ and } l_r \equiv \frac{\partial l(y, r)}{\partial r} < 0. \\
m^s &= m_o^s \text{ (i.e. the money supply is exogenous)} \\
m^s &= m^d
\end{aligned}
$$

Problem 34 *The IS curve is defined by* $y = c(y, r) + i_o + g_o$. *Show how this can be written as an implicit function of the form:*

$$g_1(y, r, i_o, g_o, m_o^s) = 0.$$

Take the total differential of this implicit function and show that the IS curve is downward sloping.

Answer

We have:

$$g_1(y, r, i_o, g_o, m_o^s) = y - c(y, r) - i_0 - g_o = 0$$

so that the total differential is:

$$(1 - c_y)\, dy - c_r dr - di_0 - dg_0 = 0.$$

Setting $di_0 = dg_0 = 0$ and solving for the slope of the IS curve $\frac{dr}{dy}$ we obtain:

$$\frac{dr}{dy} = \frac{(1 - c_y)}{c_r} < 0$$

since $0 < (1 - c_y) < 1$ and $c_r < 0$.

Problem 35 *Using the total differentials show that the conditions of the implicit function theorem are satisfied by showing that the appropriate matrix is nonsingular. Use this to show that*

$$\begin{aligned} \frac{\partial y}{\partial g_o} &> 0 \text{ and} \\ \frac{\partial r}{\partial g_o} &< 0. \end{aligned}$$

Answer

Writing the total differential in matrix notation we have:

$$\begin{bmatrix} 1 - c_y & -c_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} dy \\ dr \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} di_0 \\ dg_0 \\ dm_0^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By the implicit function theorem the reduced form:

$$y = f_1(i_o, g_o, m_o^s),\, r = f_2(i_o, g_o, m_o^s)$$

exists since:

$$\det \begin{bmatrix} 1 - c_y & -c_r \\ l_y & l_r \end{bmatrix} = (1 - c_y)\, l_r + c_r l_y < 0.$$

Setting $dm_0^s = 0$ in the total differential and writing this in matrix notation we obtain:

$$\begin{bmatrix} 1 - c_y & -c_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial g_o} \\ \frac{\partial r}{\partial g_o} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that:

$$\frac{\partial y}{\partial g_o} = \frac{\det\begin{bmatrix} 1 & -c_r \\ 0 & l_r \end{bmatrix}}{\det\begin{bmatrix} 1-c_y & -c_r \\ l_y & l_r \end{bmatrix}} = \frac{l_r}{\det\begin{bmatrix} 1-c_y & -c_r \\ l_y & l_r \end{bmatrix}} > 0$$

since both the numerator and denominator are negative.

Setting $dg_0 = 0$ in the total differential and writing this in matrix notation we obtain:

$$\begin{bmatrix} 1-c_y & -c_r \\ l_y & l_r \end{bmatrix}\begin{bmatrix} \frac{\partial y}{\partial m_o^s} \\ \frac{\partial r}{\partial m_o^s} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that:

$$\frac{\partial r}{\partial m_o^s} = \frac{\det\begin{bmatrix} 1-c_y & 0 \\ l_y & 1 \end{bmatrix}}{\det\begin{bmatrix} 1-c_y & -c_r \\ l_y & l_r \end{bmatrix}} = \frac{1-c_y}{\det\begin{bmatrix} 1-c_y & -c_r \\ l_y & l_r \end{bmatrix}} < 0$$

since the numerator is positive while the denominator is negative.

Problem 36 *Show that an increase in the money supply increases consumption. Does it follow that increasing government expenditure increases consumption.*

Answer

We have:

$$c = c(y, r) \Longrightarrow \frac{\partial c(y,r)}{\partial m_o^s} = c_y \frac{\partial y}{\partial m_o^s} + c_r \frac{\partial r}{\partial m_o^s} > 0$$

since: $\frac{\partial y}{\partial m_o^s} > 0$ (from the previous problem), $c_y > 0$, $c_r < 0$ and $\frac{\partial r}{\partial m_o^s} < 0$ (from the previous problem).

For government expenditure we have:

$$c = c(y, r) \Longrightarrow \frac{\partial c(y,r)}{\partial g_o} = c_y \frac{\partial y}{\partial g_o} + c_r \frac{\partial r}{\partial g_o}.$$

We cannot here show that $\frac{\partial c(y,r)}{\partial g_o} > 0$ since while the first term is positive the second is negative since $c_r < 0$ and $\frac{\partial r}{\partial g_0} > 0$.

The following 3 problems are based on the following information:

Consider the following IS/LM model where:

$$\begin{aligned} c^d &= c(r)\ \text{, where } c_r \equiv c'(r) < 0 \\ i^d &= i(y),\ \text{where } 0 < i_y \equiv i'(y) < 1 \\ g^d &= g_o(\text{government expenditure is exogenous}) \\ y &= c^d + i^d + g^d \\ m^d &= l(y,r),\ \text{where } l_y \equiv \frac{\partial l(y,r)}{\partial y} > 0 \text{ and } l_r \equiv \frac{\partial l(y,r)}{\partial r} < 0 \\ m^s &= m_o^s \text{ (money supply is exogenous).} \\ m^s &= m^d \end{aligned}$$

Problem 37 *Show that for this model the IS curve is downward sloping and the LM curve is upward sloping. Using the total differentials show that the conditions of the implicit function theorem are satisfied by showing that the appropriate matrix is nonsingular. Use this to show that*

$$\begin{aligned} \frac{\partial y}{\partial g_o} &> 0 \textit{ and} \\ \frac{\partial r}{\partial g_o} &< 0. \end{aligned}$$

Answer

We have:

$$g_1(y,r,i_o,g_o,m_o^s) = y - c(r) - i(y) - g_o = 0$$

so that the total differential is:

$$(1 - i_y)\,dy - c_r dr - dg_0 = 0.$$

Setting $dg_0 = 0$ and solving for the slope of the IS curve $\frac{dr}{dy}$ we obtain:

$$\frac{dr}{dy} = \frac{(1 - i_y)}{c_r} < 0$$

since $0 < (1 - i_y) < 1$ and $c_r < 0$.

The LM curve is defined by the implicit function:

$$g_2(y,r,g_o,m_o^s) = l(y,r) - m_0^s = 0$$

with total differential:

$$l_y dy + l_r dr - dm_0^s = 0.$$

Setting $dm_o^s = 0$ and solving for the slope of the LM curve: $\frac{dr}{dy}$ we obtain:

$$\frac{dr}{dy} = -\frac{l_y}{l_r} > 0$$

since $l_y > 0$ and $l_r < 0$.

Writing the total differential in matrix notation we have:

$$\begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} dy \\ dr \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} dg_0 \\ dm_0^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By the implicit function theorem the reduced form:

$$y = f_1(g_o, m_o^s), \; r = f_2(g_o, m_o^s)$$

exists since:

$$\det \begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix} = (1 - i_y)\, l_r + c_r l_y < 0.$$

Setting $dm_0^s = 0$ in the total differential and writing this in matrix notation we obtain:

$$\begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial g_o} \\ \frac{\partial r}{\partial g_o} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial y}{\partial g_o} &= \frac{\det \begin{bmatrix} 1 & -c_r \\ 0 & l_r \end{bmatrix}}{\det \begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix}} \\ &= \frac{l_r}{\det \begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix}} > 0 \\ \frac{\partial r}{\partial g_o} &= \frac{\det \begin{bmatrix} 1 - i_y & 1 \\ l_y & 0 \end{bmatrix}}{\det \begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix}} \\ &= \frac{-l_y}{\det \begin{bmatrix} 1 - i_y & -c_r \\ l_y & l_r \end{bmatrix}} > 0 \end{aligned}$$

since both the numerator and denominator for both $\frac{\partial y}{\partial g_o}$ and $\frac{\partial r}{\partial g_o}$ are negative.

Problem 38 *Using Cramer's rule, calculate* $\frac{\partial y}{\partial g_o}$ *and show that* $\frac{\partial y}{\partial g_o} > 0$. *Do the same thing with* $\frac{\partial r}{\partial m_o^s}$ *and show that* $\frac{\partial r}{\partial m_o^s} < 0$.

Answer

Setting $dg_0 = 0$ in the total differential and writing this in matrix notation we obtain:

$$\begin{bmatrix} 1-i_y & -c_r \\ l_y & l_r \end{bmatrix} \begin{bmatrix} \frac{\partial y}{\partial m_o^s} \\ \frac{\partial r}{\partial m_o^s} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial r}{\partial m_o^s} &= \frac{\det \begin{bmatrix} 1-i_y & 0 \\ l_y & 1 \end{bmatrix}}{\det \begin{bmatrix} 1-i_y & -c_r \\ l_y & l_r \end{bmatrix}} \\ &= \frac{1-i_y}{\det \begin{bmatrix} 1-i_y & -c_r \\ l_y & l_r \end{bmatrix}} < 0. \end{aligned}$$

1.5 Profit Maximization

Problem 39 *Consider a profit maximizing firm with a concave production function: $Q = F(L, K)$. We can write the conditions for profit maximization as*

$$\begin{aligned} \frac{\partial F(L^*, K^*)}{\partial L} &= w \\ \frac{\partial F(L^*, K^*)}{\partial K} &= r \end{aligned}$$

where w and r are the real wage and rental cost of capital. Write these two conditions as implicit functions, calculate the total differential and use this to show that $\frac{\partial L^*}{\partial w} < 0$ and $\frac{\partial L^*}{\partial r} = \frac{\partial K^*}{\partial w}$. Use the notation:

$$F_{LL} \equiv \frac{\partial^2 F(L^*, K^*)}{\partial L^2}, F_{LK} \equiv \frac{\partial^2 F(L^*, K^*)}{\partial L \partial K} F_{KK} \equiv \frac{\partial^2 F(L^*, K^*)}{\partial K^2}.$$

Answer

We have:

$$\begin{aligned} g_1(L^*, K^*, w, r) &= \frac{\partial F(L^*, K^*)}{\partial L} - w = 0 \\ g_2(L^*, K^*, w, r) &= \frac{\partial F(L^*, K^*)}{\partial K} - r = 0. \end{aligned}$$

The total differential is then:

$$\begin{aligned} \frac{\partial^2 F(L^*, K^*)}{\partial L^2} dL^* + \frac{\partial^2 F(L^*, K^*)}{\partial L \partial K} dK^* - dw &= 0 \\ \frac{\partial^2 F(L^*, K^*)}{\partial L \partial K} dL^* + \frac{\partial^2 F(L^*, K^*)}{\partial K^2} dK^* - dr &= 0 \end{aligned}$$

or:

$$\begin{aligned} F_{LL}dL^* + F_{LK}dK^* - dw &= 0 \\ F_{LK}dL^* + F_{KK}dK^* - dr &= 0 \end{aligned}$$

Setting $dr = 0$ we have:

$$\begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial w} \\ \frac{\partial K^*}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Note that the matrix on the left:

$$H = H(L^*, K^*) = \begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix}$$

is negative definite since $F(L, K)$ is concave. It follows then that $\det[H] > 0$ and the diagonal elements are negative, in particular $F_{LL} < 0$ and $F_{KK} < 0$. We find that:

$$\begin{aligned} \frac{\partial L^*}{\partial w} &= \frac{\det \begin{bmatrix} 1 & F_{LK} \\ 0 & F_{KK} \end{bmatrix}}{\det[H]} = \frac{F_{KK}}{\det[H]} < 0 \\ \frac{\partial K^*}{\partial w} &= \frac{\det \begin{bmatrix} F_{LL} & 1 \\ F_{LK} & 0 \end{bmatrix}}{\det[H(L^*, K^*)]} = \frac{-F_{LK}}{\det[H]} \end{aligned}$$

Setting $dw = 0$ we have:

$$\begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial r} \\ \frac{\partial K^*}{\partial r} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that:

$$\frac{\partial L^*}{\partial r} = \frac{\det \begin{bmatrix} 0 & F_{LK} \\ 1 & F_{KK} \end{bmatrix}}{\det[H]} = \frac{-F_{LK}}{\det[H]} = \frac{\partial K^*}{\partial w}.$$

Problem 40 *Consider the profit maximization problem where:*

$$\pi(L, K, P, W, R) = PF(L, K) - WL - RK.$$

Show that the firm's supply curve:

$$Q^*(P, W, R) = F(L^*(P, W, R), K^*(P, W, R))$$

is upward sloping (i.e., $\frac{\partial Q^*(P,W,R)}{\partial P} > 0$ *) using total differentials. Show that*

$$\frac{\partial Q^*(P, W, R)}{\partial W} = -\frac{\partial L^*(P, W, R)}{\partial P}$$

using total differentials.

Answer

We have:

$$\begin{aligned} g_1(L^*, K^*, P, W, R) &= P\frac{\partial F(L^*, K^*)}{\partial L} - W = 0 \\ g_2(L^*, K^*, P, W, R) &= P\frac{\partial F(L^*, K^*)}{\partial K} - R = 0. \end{aligned}$$

The total differential is then:

$$\begin{aligned} PF_{LL}dL^* + PF_{LK}dK^* + F_L dP - dW &= 0 \\ PF_{LK}dL^* + PF_{KK}dK^* + F_K dP - dR &= 0 \end{aligned}$$

where:

$$\begin{aligned} F_L &\equiv \frac{\partial F(L^*, K^*)}{\partial L}, F_K \equiv \frac{\partial F(L^*, K^*)}{\partial K} \\ F_{LL} &\equiv \frac{\partial^2 F(L^*, K^*)}{\partial L^2}, F_{LK} \equiv \frac{\partial^2 F(L^*, K^*)}{\partial L \partial K} F_{KK} \equiv \frac{\partial^2 F(L^*, K^*)}{\partial K^2} \end{aligned}$$

We will need :$\frac{\partial L^*}{\partial P}$ and $\frac{\partial K^*}{\partial P}$. Setting $dW = dR = 0$ we obtain:

$$\begin{bmatrix} PF_{LL} & PF_{LK} \\ PF_{LK} & PF_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial P} \\ \frac{\partial K^*}{\partial P} \end{bmatrix} = \begin{bmatrix} -F_L \\ -F_K \end{bmatrix}$$

or:

$$\begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial P} \\ \frac{\partial K^*}{\partial P} \end{bmatrix} = -\frac{1}{P} \begin{bmatrix} F_L \\ F_K \end{bmatrix}$$

so that:

$$\begin{bmatrix} \frac{\partial L^*}{\partial P} \\ \frac{\partial K^*}{\partial P} \end{bmatrix} = -\frac{1}{P} \begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix}^{-1} \begin{bmatrix} F_L \\ F_K \end{bmatrix}.$$

Note that the matrix:

$$\begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix}$$

is negative definite since $F(L, K)$ is concave and thus

$$A = \begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix}^{-1}$$

is also negative definite.

Now from:

$$Q^*(P, W, R) = F(L^*(P, W, R), K^*(P, W, R))$$

we have:

$$\begin{aligned}
\frac{\partial Q^*}{\partial P} &= \frac{\partial F(L^*,K^*)}{\partial L}\frac{\partial L^*}{\partial P} + \frac{\partial F(L^*,K^*)}{\partial K}\frac{\partial K^*}{\partial P} = F_L\frac{\partial L^*}{\partial P} + F_K\frac{\partial K^*}{\partial P} \\
&= \begin{bmatrix} F_L & F_K \end{bmatrix}\begin{bmatrix} \frac{\partial L^*}{\partial P} \\ \frac{\partial K^*}{\partial P} \end{bmatrix} \\
&= -\frac{1}{P}\begin{bmatrix} \frac{\partial F(L^*,K^*)}{\partial L} & \frac{\partial F(L^*,K^*)}{\partial K} \end{bmatrix}\begin{bmatrix} F_{LL} & F_{LK} \\ F_{LK} & F_{KK} \end{bmatrix}^{-1}\begin{bmatrix} F_L \\ F_K \end{bmatrix} \\
&= -\frac{1}{P}x^T Ax
\end{aligned}$$

where:

$$x = \begin{bmatrix} F_L \\ F_K \end{bmatrix} \neq 0$$

since $F_L > 0$ and $F_K > 0$. Since A is negative definite and $x \neq 0$ it follows that: $x^T Ax < 0$ and so:

$$\frac{\partial Q^*}{\partial P} = -\frac{1}{P}x^T Ax > 0$$

and the supply curve slopes upwards.

We now show that $\frac{\partial Q^*(P,W,R)}{\partial W} = -\frac{\partial L^*(P,W,R)}{\partial P}$. Using the total differential and calculating $\frac{\partial L^*(P,W,R)}{\partial P}$ first from:

$$\begin{bmatrix} PF_{LL} & PF_{LK} \\ PF_{LK} & PF_{KK} \end{bmatrix}\begin{bmatrix} \frac{\partial L^*}{\partial P} \\ \frac{\partial K^*}{\partial P} \end{bmatrix} = \begin{bmatrix} -F_L \\ -F_K \end{bmatrix}$$

and defining:

$$H = \begin{bmatrix} PF_{LL} & PF_{LK} \\ PF_{LK} & PF_{KK} \end{bmatrix}$$

we find that:

$$\begin{aligned}
\frac{\partial L^*}{\partial P} &= \frac{\det\begin{bmatrix} -F_L & PF_{LK} \\ -F_K & PF_{KK} \end{bmatrix}}{\det[H]} \\
&= \frac{-PF_LF_{KK} + PF_KF_{LK}}{\det[H]}.
\end{aligned}$$

Now we calculate $\frac{\partial Q^*(P,W,R)}{\partial W}$ from:

$$Q^*(P,W,R) = F(L^*(P,W,R), K^*(P,W,R))$$

using the chain rule it then follows that:

$$\begin{aligned}
\frac{\partial Q^*}{\partial W} &= \frac{\partial F(L^*,K^*)}{\partial L}\frac{\partial L^*}{\partial W} + \frac{\partial F(L^*,K^*)}{\partial K}\frac{\partial K^*}{\partial W} \\
&= F_L\frac{\partial L^*}{\partial W} + F_K\frac{\partial K^*}{\partial W} \\
&= \begin{bmatrix} F_L & F_K \end{bmatrix}\begin{bmatrix} \frac{\partial L^*}{\partial W} \\ \frac{\partial K^*}{\partial W} \end{bmatrix}
\end{aligned}$$

Now from:

$$\begin{bmatrix} PF_{LL} & PF_{LK} \\ PF_{LK} & PF_{KK} \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial W} \\ \frac{\partial K^*}{\partial W} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have:

$$\frac{\partial L^*}{\partial W} = \frac{PF_{KK}}{\det[H]}, \frac{\partial K^*}{\partial W} = -\frac{PF_{LK}}{\det[H]}$$

so that:

$$\begin{aligned} \frac{\partial Q^*}{\partial W} &= \begin{bmatrix} F_L & F_K \end{bmatrix} \begin{bmatrix} \frac{\partial L^*}{\partial W} \\ \frac{\partial K^*}{\partial W} \end{bmatrix} \\ &= \begin{bmatrix} F_L & F_K \end{bmatrix} \begin{bmatrix} \frac{PF_{KK}}{\det[H]} \\ -\frac{PF_{LK}}{\det[H]} \end{bmatrix} \\ &= F_L \frac{PF_{KK}}{\det[H]} - F_K \frac{PF_{LK}}{\det[H]} \\ &= -\left(\frac{-PF_L F_{KK} + PF_K F_{LK}}{\det[H]} \right) \\ &= -\frac{\partial L^*}{\partial P}. \end{aligned}$$

The following 5 problems are based on the following information:

Consider the Cobb-Douglas production function:

$$Q = F(L, K) = L^{\frac{1}{2}} K^{\frac{1}{3}}$$

which is homogeneous of degree $\frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1$. The real profit function $\pi(L, K, W, R)$ is then given by:

$$\pi(L, K, W, R) = L^{\frac{1}{2}} K^{\frac{1}{3}} - wL - rK$$

where w is the real wage and r is the real rental cost of capital.

Problem 41 *Find the first-order conditions for maximizing profits and the profit maximizing levels of labour and capital L^* and K^*. Let $l^* = \ln(L^*)$ and $k^* = \ln(K^*)$. From the first-order conditions show that:*

$$\begin{bmatrix} \frac{1}{2} - 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} - 1 \end{bmatrix} \begin{bmatrix} l^* \\ k^* \end{bmatrix} = \begin{bmatrix} \ln(2w) \\ \ln(3r) \end{bmatrix}$$

Answer

The first-order conditions are:

$$\begin{aligned}
\frac{1}{2}(L^*)^{\frac{1}{2}-1}(K^*)^{\frac{1}{3}} - w &= 0 \Longrightarrow \left(\frac{1}{2}-1\right)l^* + \frac{1}{3}k^* = \ln(2w) \\
\frac{1}{3}(L^*)^{\frac{1}{2}}(K^*)^{\frac{1}{3}-1} - r &= 0 \Longrightarrow \frac{1}{2}l^* + \left(\frac{1}{3}-1\right)k^* = \ln(3r)
\end{aligned}$$

so that writing this in matrix notation the desired result follows.

Problem 42 *From the matrix expression in find $l^*(W,R)$ and $k^*(W,R)$ using Cramer's rule. From these expressions show that the elasticities of demand for labour and capital are negative or that*

$$\begin{aligned}
\frac{\partial l^*(w,r)}{\partial \ln(w)} &= \frac{\frac{1}{3}-1}{1-(\frac{1}{2}+\frac{1}{3})} < 0 \textit{ and} \\
\frac{\partial k^*(w,r)}{\partial \ln(w)} &= \frac{\frac{1}{2}-1}{1-(\frac{1}{2}+\frac{1}{3})} < 0.
\end{aligned}$$

Answer

We have:

$$\det\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3}-1 \end{bmatrix} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

so that using Cramer's rule:

$$\begin{aligned}
l^*(w,r) &= \frac{\det\begin{bmatrix} \ln(2w) & \frac{1}{3} \\ \ln(3r) & \frac{1}{3}-1 \end{bmatrix}}{\det\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3}-1 \end{bmatrix}} = -4\ln(2w) - 2\ln(3r) \\
k^*(w,r) &= \frac{\det\begin{bmatrix} \frac{1}{2}-1 & \ln(2w) \\ \frac{1}{2} & \ln(3r) \end{bmatrix}}{\det\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3}-1 \end{bmatrix}} = -3\ln(3r) - 3\ln(2w).
\end{aligned}$$

It follows then that:

$$\begin{aligned}
\frac{\partial l^*(w,r)}{\partial \ln(w)} &= -4 < 0 \text{ and} \\
\frac{\partial k^*(w,r)}{\partial \ln(w)} &= -3 < 0.
\end{aligned}$$

Problem 43 *Find $L^*(w,r)$ and $K^*(w,r)$ from the expressions for $l^*(w,r)$ and $k^*(w,r)$ you found above. Verify from these that:*

$$\frac{\partial L^*(w,r)}{\partial r} = \frac{\partial K^*(w,r)}{\partial w}.$$

Answer

We have:

$$\begin{aligned} L^*(w,r) &= \exp\left(l^*(w,r)\right) = \frac{1}{144} w^{-4} r^{-2} \\ K^*(w,r) &= \exp\left(k^*(w,r)\right) = \frac{1}{216} w^{-3} r^{-3} \end{aligned}$$

from which:

$$\begin{aligned} \frac{\partial L^*(w,r)}{\partial r} &= -\frac{1}{72} w^{-4} r^{-3} \\ \frac{\partial K^*(w,r)}{\partial w} &= -\frac{1}{72} w^{-4} r^{-3} \end{aligned}$$

and so they are both equal to each other.

Problem 44 *Show that the second-order conditions for profit maximization are satisfied.*

Answer

The Hessian of $\pi(L,K,w,r)$ with respect to L and K is:

$$\begin{aligned} H(L,K) &= \begin{bmatrix} \frac{\partial^2 \pi}{\partial L^2} & \frac{\partial^2 \pi}{\partial L \partial K} \\ \frac{\partial^2 \pi}{\partial L \partial K} & \frac{\partial^2 \pi}{\partial K^2} \end{bmatrix} \\ &= P \begin{bmatrix} -\frac{1}{4} L^{-\frac{3}{2}} K^{\frac{1}{3}} & \frac{1}{6} L^{-\frac{1}{2}} K^{-\frac{2}{3}} \\ \frac{1}{6} L^{-\frac{1}{2}} K^{-\frac{2}{3}} & -\frac{2}{9} L^{\frac{1}{2}} K^{-\frac{5}{3}} \end{bmatrix} \end{aligned}$$

so that using leading principal minors:

$$\begin{aligned} M_1 &= P\left(-\frac{1}{4} L^{-\frac{3}{2}} K^{\frac{1}{3}}\right) < 0 \\ M_2 &= \det\left(P \begin{bmatrix} -\frac{1}{4} L^{-\frac{3}{2}} K^{\frac{1}{3}} & \frac{1}{6} L^{-\frac{1}{2}} K^{-\frac{2}{3}} \\ \frac{1}{6} L^{-\frac{1}{2}} K^{-\frac{2}{3}} & -\frac{2}{9} L^{\frac{1}{2}} K^{-\frac{5}{3}} \end{bmatrix}\right) \\ &= \frac{1}{36} P^2 L^{2\frac{1}{2}-2} K^{-\frac{4}{3}} > 0. \end{aligned}$$

Problem 45 *Show that the profit function*

$$\pi^*(w,r) = \pi(L^*(w,r), K^*(w,r), w, r)$$

is given by:

$$\pi^*(w,r) = \frac{1}{432} w^{-3} r^{-2}.$$

From this show that:

$$\begin{aligned} \frac{\partial \pi^*(w,r)}{\partial w} &= -L^*(w,r) \text{ and} \\ \frac{\partial \pi^*(w,r)}{\partial r} &= -K^*(w,r). \end{aligned}$$

Answer

We have shown that:

$$\begin{aligned} L^*(w,r) &= \exp\left(l^*(w,r)\right) = \frac{1}{144}w^{-4}r^{-2} \\ K^*(w,r) &= \exp\left(k^*(w,r)\right) = \frac{1}{216}w^{-3}r^{-3} \end{aligned}$$

so that

$$\begin{aligned} \pi^*(w,r) &= (L^*)^{\frac{1}{2}}(K^*)^{\frac{1}{3}} - wL^* - rK^* \\ &= \left(\frac{1}{144}w^{-4}r^{-2}\right)^{\frac{1}{2}}\left(\frac{1}{216}w^{-3}r^{-3}\right)^{\frac{1}{3}} - w\frac{1}{144}w^{-4}r^{-2} - r\frac{1}{216}w^{-3}r^{-3} \\ &= \frac{1}{432}w^{-3}r^{-2}. \end{aligned}$$

It follows then that:

$$\begin{aligned} \frac{\partial \pi^*(w,r)}{\partial w} &= -3 \times \frac{1}{432}w^{-4}r^{-2} \\ &= -\frac{1}{144}w^{-4}r^{-2} = -L^*(w,r). \\ \frac{\partial \pi^*(w,r)}{\partial r} &= -2 \times \frac{1}{432}w^{-3}r^{-3} \\ &= -\frac{1}{216}w^{-3}r^{-3} = -K^*(w,r). \end{aligned}$$

The following 5 problems are based on the following information:

Consider the Cobb-Douglas production function:

$$Q = F(L,K) = L^{\alpha}K^{\beta}, \text{ with } \alpha + \beta < 1.$$

This generalizes the previous set of problems by replacing $\frac{1}{2}$ with α and $\frac{1}{3}$ with β. The real profit function $\pi(L,K,W,R)$ is then given by:

$$\pi(L,K,W,R) = L^{\alpha}K^{\beta} - wL - rK$$

where w is the real wage and r is the real rental cost of capital.

Problem 46 *Find the first-order conditions for maximizing profits and the profit maximizing levels of labour and capital L^* and K^*. Let $l^* = \ln(L^*)$ and $k^* = \ln(K^*)$. From the first-order conditions show that:*

$$\begin{bmatrix} \alpha - 1 & \beta \\ \alpha & \beta - 1 \end{bmatrix} \begin{bmatrix} l^* \\ k^* \end{bmatrix} = \begin{bmatrix} \ln(w/\alpha) \\ \ln(r/\beta) \end{bmatrix}$$

Answer

The first-order conditions are:

$$\begin{aligned}\alpha\left(L^*\right)^{\alpha-1}\left(K^*\right)^{\beta}-w &= 0 \Longrightarrow (\alpha-1)\,l^*+\beta k^*=\ln(w/\alpha)\\ \beta\left(L^*\right)^{\alpha}\left(K^*\right)^{\beta-1}-r &= 0 \Longrightarrow \alpha l^*+(\beta-1)\,k^*=\ln(r/\beta)\end{aligned}$$

so that writing this in matrix notation the desired result follows.

Problem 47 *From the matrix expression in find $l^*(W,R)$ and $k^*(W,R)$ using Cramer's rule. From these expressions show that the elasticities of demand for labour and capital are negative or that*

$$\begin{aligned}\frac{\partial l^*(w,r)}{\partial\ln(w)} &= \frac{\beta-1}{1-(\alpha+\beta)}<0 \textit{ and}\\ \frac{\partial k^*(w,r)}{\partial\ln(w)} &= \frac{\alpha-1}{1-(\alpha+\beta)}<0.\end{aligned}$$

Answer

We have:

$$\det\begin{bmatrix}\alpha-1 & \beta\\ \alpha & \beta-1\end{bmatrix}=1-\alpha-\beta$$

so that using Cramer's rule:

$$\begin{aligned}l^*(w,r) &= \frac{\det\begin{bmatrix}\ln(w/\alpha) & \beta\\ \ln(r/\beta) & \beta-1\end{bmatrix}}{\det\begin{bmatrix}\alpha-1 & \beta\\ \alpha & \beta-1\end{bmatrix}}=\frac{\beta-1}{1-(\alpha+\beta)}\ln(w/\alpha)-\frac{\beta}{1-(\alpha+\beta)}\ln(r/\beta)\\ k^*(w,r) &= \frac{\det\begin{bmatrix}\alpha-1 & \ln(w/\alpha)\\ \alpha & \ln(r/\beta)\end{bmatrix}}{\det\begin{bmatrix}\alpha-1 & \beta\\ \alpha & \beta-1\end{bmatrix}}=\frac{\alpha-1}{1-(\alpha+\beta)}\ln(r/\beta)-\frac{\alpha}{1-(\alpha+\beta)}\ln(w/\alpha).\end{aligned}$$

Since $\ln(w/\alpha)=\ln(w)-\ln(\alpha)$, $\ln(r/\beta)=\ln(r)-\ln(\beta)$, $1-(\alpha+\beta)>0$, $\beta-1<0$ and $\alpha-1<0$ it follows then that:

$$\begin{aligned}\frac{\partial l^*(w,r)}{\partial\ln(w)} &= \frac{\beta-1}{1-(\alpha+\beta)}<0 \text{ and}\\ \frac{\partial k^*(w,r)}{\partial\ln(w)} &= \frac{\alpha-1}{1-(\alpha+\beta)}<0.\end{aligned}$$

Problem 48 *Find $L^*(w,r)$ and $K^*(w,r)$ from the expressions for $l^*(w,r)$ and $k^*(w,r)$ you found above. Verify from these that:*

$$\frac{\partial L^*(w,r)}{\partial r}=\frac{\partial K^*(w,r)}{\partial w}.$$

Answer

We have:

$$\begin{aligned} L^*(w,r) &= \exp\left(l^*(w,r)\right) = \left(\frac{w}{\alpha}\right)^{\frac{\beta-1}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{-\frac{\beta}{1-(\alpha+\beta)}} \\ K^*(w,r) &= \exp\left(k^*(w,r)\right) = \left(\frac{w}{\alpha}\right)^{-\frac{\alpha}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{\frac{\alpha-1}{1-(\alpha+\beta)}} \end{aligned}$$

from which:

$$\begin{aligned} \frac{\partial L^*(w,r)}{\partial r} &= \left(-\frac{\beta}{1-(\alpha+\beta)}\frac{1}{\beta}\right)\left(\frac{w}{\alpha}\right)^{\frac{\beta-1}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{-\frac{\beta}{1-(\alpha+\beta)}-1} \\ &= -\left(\frac{1}{1-(\alpha+\beta)}\right)\left(\frac{w}{\alpha}\right)^{\frac{\beta-1}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{-\frac{\beta}{1-(\alpha+\beta)}-\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)}} \\ &= -\left(\frac{1}{1-(\alpha+\beta)}\right)\left(\frac{w}{\alpha}\right)^{\frac{\beta-1}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{\frac{\alpha-1}{1-(\alpha+\beta)}} \\ \frac{\partial K^*(w,r)}{\partial w} &= \left(-\frac{\alpha}{1-(\alpha+\beta)}\frac{1}{\alpha}\right)\left(\frac{w}{\alpha}\right)^{-\frac{\alpha}{1-(\alpha+\beta)}-1}\left(\frac{r}{\beta}\right)^{\frac{\alpha-1}{1-(\alpha+\beta)}} \\ &= -\left(\frac{1}{1-(\alpha+\beta)}\right)\left(\frac{w}{\alpha}\right)^{-\frac{\alpha}{1-(\alpha+\beta)}-\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{\frac{\alpha-1}{1-(\alpha+\beta)}} \\ &= -\left(\frac{1}{1-(\alpha+\beta)}\right)\left(\frac{w}{\alpha}\right)^{\frac{\beta-1}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{\frac{\alpha-1}{1-(\alpha+\beta)}} \end{aligned}$$

and so they are both equal to each other.

Problem 49 *Show that the second-order conditions for profit maximization are satisfied given that* $\alpha+\beta<1$.

Answer

The Hessian of $\pi(L,K,w,r)$ with respect to L and K is:

$$\begin{aligned} H(L,K) &= \begin{bmatrix} \frac{\partial^2\pi}{\partial L^2} & \frac{\partial^2\pi}{\partial L\partial K} \\ \frac{\partial^2\pi}{\partial L\partial K} & \frac{\partial^2\pi}{\partial K^2} \end{bmatrix} \\ &= P\begin{bmatrix} \alpha(\alpha-1)L^{\alpha-2}K^{\beta} & \alpha\beta L^{\alpha-1}K^{\beta-1} \\ \alpha\beta L^{\alpha-1}K^{\beta-1} & \beta(\beta-1)L^{\alpha}K^{\beta-2} \end{bmatrix} \end{aligned}$$

so that using leading principal minors:

$$\begin{aligned} M_1 &= P\alpha(\alpha-1)L^{\alpha-2}K^{\beta} < 0 \text{ (since } 0<\alpha<1) \\ M_2 &= \det\left(P\begin{bmatrix} \alpha(\alpha-1)L^{\alpha-2}K^{\beta} & \alpha\beta L^{\alpha-1}K^{\beta-1} \\ \alpha\beta L^{\alpha-1}K^{\beta-1} & \beta(\beta-1)L^{\alpha}K^{\beta-2}\end{bmatrix}\right) \\ &= P^2L^{2\alpha-2}K^{2\beta-2}\left((\alpha(\alpha-1))\beta(\beta-1)-(\alpha\beta)^2\right) \\ &= \alpha\beta P^2L^{2\alpha-2}K^{2\beta-2}(1-\alpha-\beta) > 0 \text{ (since } \alpha+\beta<1). \end{aligned}$$

Problem 50 *Show that the profit function*

$$\pi^*(w,r) = \pi(L^*(w,r), K^*(w,r), w, r)$$

is given by:

$$\pi^*(w,r) = \left((1-(\alpha+\beta))\,\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}\right)w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}}.$$

From this show that:

$$\begin{aligned} \frac{\partial\pi^*(w,r)}{\partial w} &= -L^*(w,r) \text{ and} \\ \frac{\partial\pi^*(w,r)}{\partial r} &= -K^*(w,r). \end{aligned}$$

Answer

Simplifying we have:

$$\begin{aligned} L^*(w,r) &= \left(\frac{w}{\alpha}\right)^{\frac{\beta-1}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{-\frac{\beta}{1-(\alpha+\beta)}} \\ &= \alpha^{\frac{1-\beta}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{\beta-1}{1-(\alpha+\beta)}}r^{-\frac{\beta}{1-(\alpha+\beta)}} \\ K^*(w,r) &= \left(\frac{w}{\alpha}\right)^{-\frac{\alpha}{1-(\alpha+\beta)}}\left(\frac{r}{\beta}\right)^{\frac{\alpha-1}{1-(\alpha+\beta)}} \\ &= \alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{1-\alpha}{1-(\alpha+\beta)}}w^{-\frac{\alpha}{1-(\alpha+\beta)}}r^{\frac{\alpha-1}{1-(\alpha+\beta)}} \end{aligned}$$

so that

$$\begin{aligned} \pi^*(w,r) &= (L^*)^{\alpha}(K^*)^{\beta} - wL^* - rK^* \\ &= \left(\alpha^{\frac{1-\beta}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{\beta-1}{1-(\alpha+\beta)}}r^{-\frac{\beta}{1-(\alpha+\beta)}}\right)^{\alpha}\left(\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{1-\alpha}{1-(\alpha+\beta)}}w^{-\frac{\alpha}{1-(\alpha+\beta)}}r^{\frac{\alpha-1}{1-(\alpha+\beta)}}\right)^{\beta} \\ &\quad -w\alpha^{\frac{1-\beta}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{\beta-1}{1-(\alpha+\beta)}}r^{-\frac{\beta}{1-(\alpha+\beta)}} - r\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{1-\alpha}{1-(\alpha+\beta)}}w^{-\frac{\alpha}{1-(\alpha+\beta)}}r^{\frac{\alpha-1}{1-(\alpha+\beta)}} \\ &= \alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}} \\ &\quad -\alpha^{\frac{1-\beta}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}} - \alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{1-\alpha}{1-(\alpha+\beta)}}w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}} \\ &= w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}}\left(\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}} - \alpha^{\frac{1-\beta}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}} - \alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{1-\alpha}{1-(\alpha+\beta)}}\right) \\ &= \alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}\left(1-\alpha^{\frac{1-\beta-\alpha}{1-(\alpha+\beta)}}-\beta^{\frac{1-\alpha-\beta}{1-(\alpha+\beta)}}\right)w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}} \\ &= \alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}(1-\alpha-\beta)\,w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}}. \end{aligned}$$

It follows then that:

$$\begin{aligned}
\frac{\partial \pi^*(w,r)}{\partial w} &= \frac{-\alpha}{1-(\alpha+\beta)}\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}(1-\alpha-\beta)\,w^{\frac{-\alpha}{1-(\alpha+\beta)}-1}r^{\frac{-\beta}{1-(\alpha+\beta)}} \\
&= -\alpha^{\frac{\alpha}{1-(\alpha+\beta)}+1}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{-\alpha}{1-(\alpha+\beta)}-1}r^{\frac{-\beta}{1-(\alpha+\beta)}} \\
&= -\alpha^{\frac{1-\beta}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}w^{\frac{\beta-1}{1-(\alpha+\beta)}}r^{-\frac{\beta}{1-(\alpha+\beta)}} = -L^*(w,r) \\
\frac{\partial \pi^*(w,r)}{\partial r} &= \frac{-\beta}{1-(\alpha+\beta)}\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}}(1-\alpha-\beta)\,w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}-1} \\
&= -\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{\beta}{1-(\alpha+\beta)}+1}w^{\frac{-\alpha}{1-(\alpha+\beta)}}r^{\frac{-\beta}{1-(\alpha+\beta)}-1} \\
&= -\alpha^{\frac{\alpha}{1-(\alpha+\beta)}}\beta^{\frac{1-\alpha}{1-(\alpha+\beta)}}w^{-\frac{\alpha}{1-(\alpha+\beta)}}r^{\frac{\alpha-1}{1-(\alpha+\beta)}} = -K^*(w,r).
\end{aligned}$$

1.6 Utility Maximization

Consider a household with a utility function:

$$U(Q_1, Q_2)$$

where:

$$\begin{aligned}
U_1 &= \frac{\partial U(Q_1,Q_2)}{\partial Q_1} > 0,\ U_2 = \frac{\partial U(Q_1,Q_2)}{\partial Q_2} > 0 \\
U_{11} &= \frac{\partial^2 U(Q_1,Q_2)}{\partial Q_1^2},\ U_{22} = \frac{\partial^2 U(Q_1,Q_2)}{\partial Q_2^2},\ U_{12} = \frac{\partial^2 U(Q_1,Q_2)}{\partial Q_1 \partial Q_2}
\end{aligned}$$

and where the utility function is concave so that the Hessian:

$$H(Q_1,Q_2) = \begin{bmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{bmatrix}$$

is negative definite.

Problem 51 *Construct the Lagrangian for the utility maximization problem and derive the total differential from the first-order conditions.*

Answer

The Lagrangian for the constrained maximization problem is:

$$\mathcal{L}(\lambda, Q_1,\ Q_2) = U(Q_1,Q_2) + \lambda(Y - P_1Q_1 - P_2Q_2)$$

which yields the first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*)}{\partial \lambda} &= Y - P_1Q_1^* - P_2Q_2^* = 0 \\
\frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*)}{\partial Q_1} &= \frac{\partial U(Q_1^*,Q_2^*)}{\partial Q_1} - \lambda^* P_1 = 0 \\
\frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*)}{\partial Q_2} &= \frac{\partial U(Q_1^*,Q_2^*)}{\partial Q_2} - \lambda^* P_2 = 0.
\end{aligned}$$

Here λ^*, Q_1^* and Q_2^* are the endogenous variables while Y, P_1 and P_2 are the exogenous variables. This leads to 3 implicit functions:

$$\begin{aligned} g_1\left(\lambda^*, Q_1^*, Q_2^*, Y, P_1, P_2\right) &= Y - P_1 Q_1^* - P_2 Q_2^* = 0 \\ g_2\left(\lambda^*, Q_1^*, Q_2^*, Y, P_1, P_2\right) &= \frac{\partial U\left(Q_1^*, Q_2^*\right)}{\partial Q_1} - \lambda^* P_1 = 0 \\ g_3\left(\lambda^*, Q_1^*, Q_2^*, Y, P_1, P_2\right) &= \frac{\partial U\left(Q_1^*, Q_2^*\right)}{\partial Q_2} - \lambda^* P_2 = 0 \end{aligned}$$

which determines the reduced form:

$$\begin{aligned} \lambda^* &= \lambda^*\left(Y, P_1, P_2\right), \\ Q_1^* &= Q_1^*\left(Y, P_1, P_2\right), \\ Q_2^* &= Q_2^*\left(Y, P_1, P_2\right). \end{aligned}$$

Note that $\lambda^* > 0$ since from the second order conditions:

$$\lambda^* = \frac{1}{P_1}\frac{\partial U\left(Q_1^*, Q_2^*\right)}{\partial Q_1} = \frac{1}{P_2}\frac{\partial U\left(Q_1^*, Q_2^*\right)}{\partial Q_2} > 0.$$

Taking the total differential we find that:

$$\begin{aligned} 0d\lambda^* - P_1 dQ_1^* - P_2 dQ_2^* + dY - Q_1^* dP_1 - Q_2^* dP_2 &= 0 \\ -P_1 d\lambda^* + U_{11} dQ_1^* + U_{12} dQ_2^* - \lambda^* dP_1 &= 0 \\ -P_2 d\lambda^* + U_{12} dQ_1^* + U_{22} dQ_2^* - \lambda^* dP_2 &= 0. \end{aligned}$$

Problem 52 *What are the second-order conditions and show that they will be satisfied.*

Answer

The second-order condition for this problem is that the Hessian of $\mathcal{L}\left(\lambda^*, Q_1^*, Q_2^*\right)$ given by:

$$H = \begin{bmatrix} 0 & -P_1 & -P_2 \\ -P_1 & U_{11} & U_{12} \\ -P_2 & U_{12} & U_{22} \end{bmatrix}$$

should have a positive determinant. Now

$$\begin{aligned} \det[H] &= \det \begin{bmatrix} 0 & -P_1 & -P_2 \\ -P_1 & U_{11} & U_{12} \\ -P_2 & U_{12} & U_{22} \end{bmatrix} = -P_1^2 U_{22} + 2P_1 P_2 U_{12} - P_2^2 U_{11} \\ &= -\begin{bmatrix} P_2 & -P_1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{22} & U_{22} \end{bmatrix} \begin{bmatrix} P_2 \\ -P_1 \end{bmatrix} > 0 \end{aligned}$$

since the latter expression is of the form $-x^T H x$ with $x \neq 0$ and H is negative definite.

Problem 53 *Assume that $U_{12} > 0$. Show from the total differential that Q_1 is a normal good, that the demand curve for Q_1 is downward sloping and that there is diminishing marginal utility in income or $\frac{\partial \lambda^*}{\partial Y} < 0$.*

Answer

From the total differential with $dP_1 = dP_2 = 0$ we have:

$$\underbrace{\begin{bmatrix} 0 & -P_1 & -P_2 \\ -P_1 & U_{11} & U_{12} \\ -P_2 & U_{12} & U_{22} \end{bmatrix}}_{=H} \begin{bmatrix} \frac{\partial \lambda^*}{\partial Y} \\ \frac{\partial Q_1^*}{\partial Y} \\ \frac{\partial Q_2^*}{\partial Y} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

so that:

$$\frac{\partial Q_1^*}{\partial Y} = \frac{\det \begin{bmatrix} 0 & -1 & -P_2 \\ -P_1 & 0 & U_{12} \\ -P_2 & 0 & U_{22} \end{bmatrix}}{\det [H]} = \frac{-P_1 U_{22} + P_2 U_{12}}{\det [H]} > 0$$

since $\det [H] > 0$ and $U_{22} < 0$ since the utility function is concave and by assumption: $U_{12} > 0$.

We have for $\frac{\partial \lambda^*}{\partial Y}$ then that:

$$\frac{\partial \lambda^*}{\partial Y} = \frac{\det \begin{bmatrix} -1 & -P_1 & -P_2 \\ 0 & U_{11} & U_{12} \\ 0 & U_{12} & U_{22} \end{bmatrix}}{\det [H]} = \frac{-\det \begin{bmatrix} U_{11} & U_{12} \\ U_{12} & U_{22} \end{bmatrix}}{\det [H]} < 0$$

since $\det [H] > 0$ and the determinant in the numerator is positive by the concavity of the utility function.

From the total differential with $dP_2 = dY = 0$ we have:

$$\begin{bmatrix} 0 & -P_1 & -P_2 \\ -P_1 & U_{11} & U_{12} \\ -P_2 & U_{12} & U_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda^*}{\partial P_1} \\ \frac{\partial Q_1^*}{\partial P_1} \\ \frac{\partial Q_2^*}{\partial P_1} \end{bmatrix} = \begin{bmatrix} Q_1^* \\ \lambda^* \\ 0 \end{bmatrix}$$

so that:

$$\begin{aligned} \frac{\partial Q_1^*}{\partial P_1} &= \frac{\det \begin{bmatrix} 0 & Q_1^* & -P_2 \\ -P_1 & \lambda^* & U_{12} \\ -P_2 & 0 & U_{22} \end{bmatrix}}{\det [H]} \\ &= -Q_1^* \left(\frac{-P_1 U_{22} + P_2 U_{12}}{\det [H]} \right) + \frac{\lambda^* P_2^2 U_{22}}{\det [H]} \\ &= -Q_1^* \frac{\partial Q_1^*}{\partial Y} + \frac{\lambda^* P_2^2 U_{22}}{\det [H]} < 0 \end{aligned}$$

since $\det [H] > 0$, we have already shown that $\frac{\partial Q_1^*}{\partial Y} > 0$, $\lambda^* > 0$ and $U_{22} < 0$.

1.7 Cost Minimization

The following 2 problems are based on the following information:

Consider the problem of choosing the values of two factors of production (say labour and capital): L and K which minimize the cost C given by:

$$C = WL + RK$$

subject to the constraint that Q units are produced or that:

$$Q = L^{\frac{1}{2}} K^{\frac{1}{3}}$$

where $F(L, K)$ is the production function. The Lagrangian for this problem is then given by:

$$\mathcal{L}(\lambda, L, K, Q, W, R) = WL + RK + \lambda \left(Q - L^{\frac{1}{2}} K^{\frac{1}{3}} \right)$$

Problem 54 *Find the first-order conditions for this problem and solve for the reduced form using the* $\ln(\ )$ *function to convert the first-order conditions into a system of linear equations. To simplify your calculations you can accept (or verify if you wish) that:*

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{2}{3} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{2}{5} \\ \frac{6}{5} & -\frac{2}{5} & \frac{2}{5} \\ \frac{6}{5} & \frac{3}{5} & -\frac{3}{5} \end{bmatrix}.$$

Answer

We have the first-order conditions:

$$\begin{aligned} Q - L^{*\frac{1}{2}} K^{*\frac{1}{3}} &= 0 \\ W - \frac{1}{2}\lambda^* L^{*-\frac{1}{2}} K^{*\frac{1}{3}} &= 0 \\ R - \frac{1}{3}\lambda^* L^{*\frac{1}{2}} K^{*-\frac{2}{3}} &= 0 \end{aligned}$$

so that:

$$\begin{aligned} Q - L^{*\frac{1}{2}} K^{*\frac{1}{3}} &= 0 \Longrightarrow \frac{1}{2}\ln(L^*) + \frac{1}{3}\ln(K^*) = \ln(Q) \\ W - \frac{1}{2}\lambda^* L^{*\frac{1}{2}-1} K^{*\frac{1}{3}} &= 0 \Longrightarrow \ln(\lambda^*) + -\frac{1}{2}\ln(L^*) + \frac{1}{3}\ln(K^*) = \ln(2W) \\ R - \frac{1}{3}\lambda^* L^{*\frac{1}{2}} K^{*\frac{1}{3}-1} &= 0 \Longrightarrow \ln(\lambda^*) + \frac{1}{2}\ln(L^*) - \frac{2}{3}\ln(K^*) = \ln(3R). \end{aligned}$$

so that in matrix notation we have:

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} \ln(\lambda^*) \\ \ln(L^*) \\ \ln(K^*) \end{bmatrix} = \begin{bmatrix} \ln(Q) \\ \ln(2W) \\ \ln(3R) \end{bmatrix}.$$

Thus:

$$\begin{aligned}
\begin{bmatrix} \ln(\lambda^*) \\ \ln(L^*) \\ \ln(K^*) \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{2}{3} \end{bmatrix}^{-1} \begin{bmatrix} \ln(Q) \\ \ln(2W) \\ \ln(3R) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{2}{5} \\ \frac{6}{5} & -\frac{2}{5} & \frac{2}{5} \\ \frac{6}{5} & \frac{3}{5} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} \ln(Q) \\ \ln(2W) \\ \ln(3R) \end{bmatrix}
\end{aligned}$$

$: \begin{bmatrix} \frac{1}{5}\ln Q + \frac{3}{5}\ln 2W + \frac{2}{5}\ln 3R \\ \frac{6}{5}\ln Q - \frac{2}{5}\ln 2W + \frac{2}{5}\ln 3R \\ \frac{6}{5}\ln Q + \frac{3}{5}\ln 2W - \frac{3}{5}\ln 3R \end{bmatrix}$ or:

$$\begin{aligned}
\ln(\lambda^*) &= \frac{1}{5}\ln(Q) + \frac{3}{5}\ln(2W) + \frac{2}{5}\ln(3R) \\
\ln(L^*) &= \frac{6}{5}\ln(Q) - \frac{2}{5}\ln(2W) + \frac{2}{5}\ln(3R) \\
\ln(K^*) &= \frac{6}{5}\ln(Q) + \frac{3}{5}\ln(2W) - \frac{3}{5}\ln(3R)
\end{aligned}$$

so that:

$$\begin{aligned}
\lambda^* &= \exp\left(\frac{1}{5}\ln(Q) + \frac{3}{5}\ln(2W) + \frac{2}{5}\ln(3R)\right) \\
&= Q^{\frac{1}{5}}(2W)^{\frac{3}{5}}(3R)^{\frac{2}{5}} \\
&= 2^{\frac{3}{5}}3^{\frac{2}{5}}Q^{\frac{1}{5}}W^{\frac{3}{5}}R^{\frac{2}{5}} \\
L^* &= \exp\left(\frac{6}{5}\ln(Q) - \frac{2}{5}\ln(2W) + \frac{2}{5}\ln(3R)\right) \\
&= Q^{\frac{6}{5}}(2W)^{-\frac{2}{5}}(3R)^{\frac{2}{5}} \\
&= 2^{-\frac{2}{5}}3^{\frac{2}{5}}Q^{\frac{6}{5}}W^{-\frac{2}{5}}R^{\frac{2}{5}} \\
K^* &= \exp\left(\frac{6}{5}\ln(Q) + \frac{3}{5}\ln(2W) - \frac{3}{5}\ln(3R)\right) \\
&= Q^{\frac{6}{5}}(2W)^{\frac{3}{5}}(3R)^{-\frac{3}{5}} \\
&= 2^{\frac{3}{5}}3^{-\frac{3}{5}}Q^{\frac{6}{5}}W^{\frac{3}{5}}R^{-\frac{3}{5}}.
\end{aligned}$$

Problem 55 *Show that the cost function for this problem is:*

$$C^*(Q,W,R) = \left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right)Q^{\frac{6}{5}}W^{\frac{3}{5}}R^{\frac{2}{5}}$$

and verify that:

$$\begin{aligned}
\frac{\partial C^*(Q,W,R)}{\partial Q} &= \lambda^*(Q,W,R) \\
\frac{\partial C^*(Q,W,R)}{\partial W} &= L^*(Q,W,R) \\
\frac{\partial C^*(Q,W,R)}{\partial R} &= K^*(Q,W,R).
\end{aligned}$$

Answer

We have:

$$\begin{aligned} C^*(Q,W,R) &\equiv W \times L^*(Q,W,R) + R \times K^*(Q,W,R) \\ &= W \times Q^{\frac{6}{5}}(2W)^{-\frac{2}{5}}(3R)^{\frac{2}{5}} \\ &\quad + R \times Q^{\frac{6}{5}}(2W)^{\frac{3}{5}}(3R)^{-\frac{3}{5}} \\ &= \left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{6}{5}} W^{\frac{3}{5}} R^{\frac{2}{5}}. \end{aligned}$$

We then have:

$$\begin{aligned} \frac{\partial C^*(Q,W,R)}{\partial Q} &= \frac{\partial}{\partial Q}\left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{6}{5}} W^{\frac{3}{5}} R^{\frac{2}{5}} \\ &= \frac{6}{5}\left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{1}{5}} W^{\frac{3}{5}} R^{\frac{2}{5}} \\ &= 2^{\frac{3}{5}}3^{\frac{2}{5}} Q^{\frac{1}{5}} W^{\frac{3}{5}} R^{\frac{2}{5}} = \lambda^*(Q,W,R) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial C^*(Q,W,R)}{\partial W} &= \frac{\partial}{\partial W}\left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{6}{5}} W^{\frac{3}{5}} R^{\frac{2}{5}} \\ &= \frac{3}{5}\left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{6}{5}} W^{-\frac{2}{5}} R^{\frac{2}{5}} \\ &= 2^{-\frac{2}{5}}3^{\frac{2}{5}} Q^{\frac{6}{5}} W^{-\frac{2}{5}} R^{\frac{2}{5}} \\ &= L^*(Q,W,R) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial C^*(Q,W,R)}{\partial W} &= \frac{\partial}{\partial R}\left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{6}{5}} W^{\frac{3}{5}} R^{\frac{2}{5}} \\ &= \frac{2}{5}\left(2^{-\frac{2}{5}}3^{\frac{2}{5}} + 2^{\frac{3}{5}}3^{-\frac{3}{5}}\right) Q^{\frac{6}{5}} W^{\frac{3}{5}} R^{-\frac{3}{5}} \\ &= 2^{\frac{3}{5}}3^{-\frac{3}{5}} Q^{\frac{6}{5}} W^{\frac{3}{5}} R^{-\frac{3}{5}} \\ &= K^*(Q,W,R). \end{aligned}$$

The following 2 problems are based on the following information:

Consider the problem of choosing the values of two factors of production (say labour and capital): L and K which minimize the cost C given by:

$$C = WL + RK$$

subject to the constraint that Q units are produced or that:

$$Q = L^{\alpha} K^{\beta}$$

where $F(L,K)$ is the production function. The Lagrangian for this problem is then given by:

$$\mathcal{L}(\lambda, L, K, Q, W, R) = WL + RK + \lambda\left(Q - L^{\alpha}K^{\beta}\right)$$

Problem 56 *Find the first-order conditions for this problem and solve for the reduced form using the* $\ln(\,)$ *function to convert the first-order conditions into a system of linear equations. To simplify your calculations you can accept (or verify if you wish) that:*

$$\begin{bmatrix} 0 & \alpha & \beta \\ 1 & \alpha-1 & \beta \\ 1 & \alpha & \beta-1 \end{bmatrix}^{-1} = \frac{1}{\alpha+\beta}\begin{bmatrix} 1-\alpha-\beta & \alpha & \beta \\ 1 & -\beta & \beta \\ 1 & \alpha & -\alpha \end{bmatrix}$$

Answer

We have the first-order conditions:

$$\begin{aligned} Q - L^{*\alpha}K^{*\beta} &= 0 \\ W - \alpha\lambda^{*}L^{*\alpha-1}K^{*\beta} &= 0 \\ R - \beta\lambda^{*}L^{*\alpha}K^{*\beta-1} &= 0 \end{aligned}$$

so that:

$$\begin{aligned} Q - L^{*\alpha}K^{*\beta} &= 0 \Longrightarrow \alpha\ln(L^{*}) + \beta\ln(K^{*}) = \ln(Q) \\ W - \alpha\lambda^{*}L^{*\alpha-1}K^{*\beta} &= 0 \Longrightarrow \ln(\lambda^{*}) + (\alpha-1)\ln(L^{*}) + \beta\ln(K^{*}) = \ln\left(\frac{W}{\alpha}\right) \\ R - \beta\lambda^{*}L^{*\alpha}K^{*\beta-1} &= 0 \Longrightarrow \ln(\lambda^{*}) + \alpha\ln(L^{*}) + (\beta-1)\ln(K^{*}) = \ln\left(\frac{R}{\beta}\right). \end{aligned}$$

so that in matrix notation we have:

$$\begin{bmatrix} 0 & \alpha & \beta \\ 1 & \alpha-1 & \beta \\ 1 & \alpha & \beta-1 \end{bmatrix}\begin{bmatrix} \ln(\lambda^{*}) \\ \ln(L^{*}) \\ \ln(K^{*}) \end{bmatrix} = \begin{bmatrix} \ln(Q) \\ \ln\left(\frac{W}{\alpha}\right) \\ \ln\left(\frac{R}{\beta}\right) \end{bmatrix}.$$

Using the adjoint and determinant we have:

$$\begin{bmatrix} 0 & \alpha & \beta \\ 1 & \alpha-1 & \beta \\ 1 & \alpha & \beta-1 \end{bmatrix}^{-1} = \frac{1}{\alpha+\beta}\begin{bmatrix} 1-\alpha-\beta & \alpha & \beta \\ 1 & -\beta & \beta \\ 1 & \alpha & -\alpha \end{bmatrix}$$

so that:

$$\begin{aligned} \begin{bmatrix} \ln(\lambda^{*}) \\ \ln(L^{*}) \\ \ln(K^{*}) \end{bmatrix} &= \begin{bmatrix} 0 & \alpha & \beta \\ 1 & \alpha-1 & \beta \\ 1 & \alpha & \beta-1 \end{bmatrix}^{-1}\begin{bmatrix} \ln(Q) \\ \ln\left(\frac{W}{\alpha}\right) \\ \ln\left(\frac{R}{\beta}\right) \end{bmatrix} \\ &= \frac{1}{\alpha+\beta}\begin{bmatrix} 1-\alpha-\beta & \alpha & \beta \\ 1 & -\beta & \beta \\ 1 & \alpha & -\alpha \end{bmatrix}\begin{bmatrix} \ln(Q) \\ \ln\left(\frac{W}{\alpha}\right) \\ \ln\left(\frac{R}{\beta}\right) \end{bmatrix} \end{aligned}$$

or:

$$\begin{aligned}
\ln(\lambda^*) &= \frac{1-\alpha-\beta}{\alpha+\beta}\ln(Q) + \frac{\alpha}{\alpha+\beta}\ln\left(\frac{W}{\alpha}\right) + \frac{\beta}{\alpha+\beta}\ln\left(\frac{R}{\beta}\right) \\
\ln(L^*) &= \frac{1}{\alpha+\beta}\ln(Q) - \frac{\beta}{\alpha+\beta}\ln\left(\frac{W}{\alpha}\right) + \frac{\beta}{\alpha+\beta}\ln\left(\frac{R}{\beta}\right) \\
\ln(K^*) &= \frac{1}{\alpha+\beta}\ln(Q) + \frac{\alpha}{\alpha+\beta}\ln\left(\frac{W}{\alpha}\right) - \frac{\alpha}{\alpha+\beta}\ln\left(\frac{R}{\beta}\right)
\end{aligned}$$

so that:

$$\begin{aligned}
\lambda^* &= \lambda^*(Q,W,R) \\
&= Q^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{W}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{R}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \\
&= \alpha^{-\frac{\alpha}{\alpha+\beta}}\beta^{-\frac{\beta}{\alpha+\beta}}Q^{\frac{1-\alpha-\beta}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}} \\
L^* &= L^*(Q,W,R) \\
&= Q^{\frac{1}{\alpha+\beta}}\left(\frac{W}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}}\left(\frac{R}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \\
&= \alpha^{\frac{\beta}{\alpha+\beta}}\beta^{-\frac{\beta}{\alpha+\beta}}Q^{\frac{1}{\alpha+\beta}}W^{-\frac{\beta}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}} \\
K^* &= K^*(Q,W,R) \\
&= Q^{\frac{1}{\alpha+\beta}}\left(\frac{W}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{R}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} \\
&= \alpha^{-\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\alpha}{\alpha+\beta}}Q^{\frac{1}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{-\frac{\alpha}{\alpha+\beta}}
\end{aligned}$$

Problem 57 *Show that the cost function for this problem is:*

$$C^*(Q,W,R) = (\alpha+\beta)\,\alpha^{-\frac{\alpha}{\alpha+\beta}}\beta^{-\frac{\beta}{\alpha+\beta}}Q^{\frac{1}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}}$$

and verify that:

$$\begin{aligned}
\frac{\partial C^*(Q,W,R)}{\partial Q} &= \lambda^*(Q,W,R) \\
\frac{\partial C^*(Q,W,R)}{\partial W} &= L^*(Q,W,R) \\
\frac{\partial C^*(Q,W,R)}{\partial R} &= K^*(Q,W,R).
\end{aligned}$$

Answer

We have:

$$\begin{aligned} C^*(Q,W,R) &\equiv W\times L^*(Q,W,R)+R\times K^*(Q,W,R) \\ &= W\times \alpha^{\frac{\beta}{\alpha+\beta}}\beta^{-\frac{\beta}{\alpha+\beta}}Q^{\frac{1}{\alpha+\beta}}W^{-\frac{\beta}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}} \\ &\quad +R\times \alpha^{-\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\alpha}{\alpha+\beta}}Q^{\frac{1}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{-\frac{\alpha}{\alpha+\beta}} \\ &= \left(\alpha^{\frac{\beta}{\alpha+\beta}}\beta^{-\frac{\beta}{\alpha+\beta}}+\alpha^{-\frac{\alpha}{\alpha+\beta}}\beta^{\frac{\alpha}{\alpha+\beta}}\right)Q^{\frac{1}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}} \\ &= \alpha^{\frac{\beta}{\alpha+\beta}}\beta^{\frac{\alpha}{\alpha+\beta}}\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)Q^{\frac{1}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}} \\ &= (\alpha+\beta)\,\alpha^{-\frac{\alpha}{\alpha+\beta}}\beta^{-\frac{\beta}{\alpha+\beta}}Q^{\frac{1}{\alpha+\beta}}W^{\frac{\alpha}{\alpha+\beta}}R^{\frac{\beta}{\alpha+\beta}}. \end{aligned}$$

1.8 Homogeneous Functions

Problem 58 *Show that the function*

$$f(x_1,x_2)=(2\sqrt{x_1}+3\sqrt{x_2})^{\frac{4}{3}}$$

is homogeneous and determine k*.*

Answer

We have:

$$\begin{aligned} f(\lambda x_1,\lambda x_2) &= \left(2\sqrt{\lambda x_1}+3\sqrt{\lambda x_2}\right)^{\frac{4}{3}} \\ &= \left(\lambda^{\frac{1}{2}}(2\sqrt{x_1}+3\sqrt{x_2})\right)^{\frac{4}{3}} \\ &= \lambda^{\frac{2}{3}}(2\sqrt{x_1}+3\sqrt{x_2})^{\frac{4}{3}} \\ &= \lambda^{\frac{2}{3}}f(x_1,x_2) \end{aligned}$$

so the function is homogenous of degree $k=\frac{2}{3}$.

Problem 59 *Show that*

$$Q=F(L,K)=5L^{\alpha}K^{\beta}$$

is homogeneous and determine k*.*

Answer

We have:

$$F(\lambda L,\lambda K)=5(\lambda L)^{\alpha}(\lambda K)^{\beta}=\lambda^{\alpha}\lambda^{\beta}5(L)^{\alpha}(K)^{\beta}=\lambda^{\alpha+\beta}5(L)^{\alpha}(K)^{\beta}=\lambda^{\alpha+\beta}F(L,K)$$

so that $F(L,K)=5(L)^{\alpha}(K)^{\beta}$ is homogeneous of degree $\alpha+\beta$.

Problem 60 *Show that if $f(x_1, x_2)$ is homogenous of degree 1 then the Hessian of $f(x_1, x_2)$ is singular for all non-zero x_1, x_2.*

Answer

If $f(x_1, x_2)$ is homogenous of degree 1 then both $\frac{\partial f(x_1,x_2)}{\partial x_1}$ and $\frac{\partial f(x_1,x_2)}{\partial x_2}$ are homogeneous of degree 0. By Euler's theorem:

$$\begin{aligned} 0\frac{\partial f(x_1, x_2)}{\partial x_1} &= \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2}x_1 + \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}x_2 = 0 \\ 0\frac{\partial f(x_1, x_2)}{\partial x_2} &= \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}x_1 + \frac{\partial^2 f(x_1, x_2)}{\partial x_2^2}x_2 = 0 \end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is of the form $Hx = 0$ where H is the Hessian. Since $x \neq 0$ it follows that H is a singular matrix.

Problem 61 *Show that:*

$$Q = F(L, K) = \left(\frac{2}{3}L^{-3} + \frac{1}{3}K^{-3}\right)^{-\frac{2}{3}}$$

is homogeneous of degree 2.

Answer

Since:

$$\begin{aligned} F(\lambda L, \lambda K) &= \left(\frac{2}{3}(\lambda L)^{-3} + \frac{1}{3}(\lambda K)^{-3}\right)^{-\frac{2}{3}} \\ &= \left(\lambda^{-3}\left(\frac{2}{3}L^{-3} + \frac{1}{3}K^{-3}\right)\right)^{-\frac{2}{3}} \\ &= \lambda^2\left(\frac{2}{3}L^{-3} + \frac{1}{3}K^{-3}\right)^{-\frac{2}{3}} \\ &= \lambda^2 F(L, K). \end{aligned}$$

Problem 62 *Show that the Constant Elasticity of Substitution or CES production function:*

$$Q = F(L, K) = (\alpha L^\rho + (1-\alpha) K^\rho)^{\frac{\gamma}{\rho}}$$

is homogeneous of degree γ.

Answer

Since:

$$\begin{aligned} F(\lambda L, \lambda K) &= \left(\alpha (\lambda L)^{\rho} + (1-\alpha)(\lambda K)^{\rho}\right)^{\frac{\gamma}{\rho}} \\ &= \left(\lambda^{\rho}\left(\alpha L^{\rho} + (1-\alpha) K^{\rho}\right)\right)^{\frac{\gamma}{\rho}} \\ &= \left(\lambda^{\rho}\right)^{\frac{\gamma}{\rho}} \left(\alpha L^{\rho} + (1-\alpha) K^{\rho}\right)^{\frac{\gamma}{\rho}} \\ &= \lambda^{\gamma} \left(\alpha L^{\rho} + (1-\alpha) K^{\rho}\right)^{\frac{\gamma}{\rho}} \\ &= \lambda^{\gamma} F(L, K). \end{aligned}$$

Problem 63 *Use Euler's theorem to show that if $Q = F(L,K)$ is homogeneous of degree k and if the firm pays labour and capital according to its marginal product so that $\frac{\partial F(L,K)}{\partial L} = \frac{W}{P}$ and $\frac{\partial F(L,K)}{\partial K} = \frac{R}{P}$, then the firm receives positive,* 0, *and negative profits according to whether $k < 1$, $k = 1$ or $k > 1$.*

Answer

By Euler's theorem:

$$kQ = \frac{\partial F(L,K)}{\partial L} L + \frac{\partial F(L,K)}{\partial K} K.$$

Since $\frac{\partial F(L,K)}{\partial L} = \frac{W}{P}$ and $\frac{\partial F(L,K)}{\partial K} = \frac{R}{P}$ we have:

$$kQ = \frac{W}{P} L + \frac{R}{P} K \Longrightarrow kPQ = WL + RK$$

and so profits π are given by:

$$\begin{aligned} \pi &= PQ - (WL + RK) = PQ - kPQ \\ &= (1-k) PQ. \end{aligned}$$

Thus if $0 < k < 1$ (there are decreasing returns to scale) then $\pi > 0$ while if $k = 1$ (constant returns to scale) then $\pi = 0$. If $k > 1$ then profits must be negative, which is indicative of the fact that increasing returns to scale are not consistent with perfect competition.

Problem 64 *Suppose that the profit function is given by:*

$$\pi^*(P, W, R) = \left(a_1 P^{\alpha} + a_2 W^{\alpha} + a_3 R^{\alpha}\right)^{\frac{1}{\alpha}}$$

where $\alpha > 0$. Show that this is homogeneous of degree 1.

Answer

We have:

$$\begin{aligned}
\pi^*(\lambda P, \lambda W, \lambda R) &= \left(a_1 (\lambda P)^\alpha + a_2 (W)^\alpha + a_3 (\lambda R)^\alpha\right)^{\frac{1}{\alpha}} \\
&= \left(\lambda^\alpha \left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)\right)^{\frac{1}{\alpha}} \\
&= \left(\lambda^\alpha\right)^{\frac{1}{\alpha}} \left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}} \\
&= \lambda \left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}} \\
&= \lambda^1 \pi^*(P, W, R)
\end{aligned}$$

so it is homogeneous of degree $k = 1$.

Chapter 2

Duality

2.1 Profit Maximization

Problem 65 *Suppose the clever profit function is given by:*

$$\pi^*(P,W,R) = 30P^{\frac{8}{3}}W^{-\frac{4}{3}}R^{-\frac{1}{3}}.$$

Using this information calculate the firm's supply curve: $Q^*(P,W,R)$ *as well as the factor demands:* $L^*(P,W,R)$ *and* $K^*(P,W,R)$.

Answer

The desired functions can be obtained from differentiating $\pi^*(P,W,R)$ using Hotelling's lemma as:

$$\begin{aligned} Q^*(P,W,R) &= \frac{\partial}{\partial P}\left(30P^{\frac{8}{3}}W^{-\frac{4}{3}}R^{-\frac{1}{3}}\right) = 80P^{\frac{5}{3}}W^{-\frac{4}{3}}R^{-\frac{1}{3}} \\ L^*(P,W,R) &= -\frac{\partial}{\partial W}\left(30P^{\frac{8}{3}}W^{-\frac{4}{3}}R^{-\frac{1}{3}}\right) = 40P^{\frac{8}{3}}W^{-\frac{7}{3}}R^{-\frac{1}{3}} \\ K^*(P,W,R) &= -\frac{\partial}{\partial R}\left(30P^{\frac{8}{3}}W^{-\frac{4}{3}}R^{-\frac{1}{3}}\right) = 10P^{\frac{8}{3}}W^{-\frac{4}{3}}R^{-\frac{4}{3}}. \end{aligned}$$

Problem 66 *Suppose that the profit function is given by:*

$$\pi^*(P,W,R) = (a_1P^\alpha + a_2W^\alpha + a_3R^\alpha)^{\frac{1}{\alpha}}$$

where $\alpha > 0$. *Find the firm's supply curve and factor demand curves, and show that* $a_1 > 0$, $a_2 < 0$ *and* $a_3 < 0$.

Answer

From Hotelling's lemma we have:

$$\begin{aligned} Q^*(P,W,R) &= \frac{\partial \pi^*(P,W,R)}{\partial P} \\ &= \frac{\partial}{\partial P}\left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}} \\ &= \left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}-1} a_1 P^{\alpha-1}. \end{aligned}$$

Since $Q^*(P,W,R) > 0$ and all terms except a_1 in $Q^*(P,W,R)$ are positive we require $a_1 > 0$. Similarly we have:

$$\begin{aligned} L^*(P,W,R) &= -\frac{\partial \pi^*(P,W,R)}{\partial W} \\ &= -\frac{\partial}{\partial W}\left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}} \\ &= -a_2 W^{\alpha-1}\left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}-1}. \end{aligned}$$

Since $L^*(P,W,R) > 0$ and all terms except $-a_2$ in $L^*(P,W,R)$ are positive we have $-a_2 > 0$ or $a_2 < 0$. Similarly we have:

$$\begin{aligned} K^*(P,W,R) &= -\frac{\partial \pi^*(P,W,R)}{\partial R} \\ &= -\frac{\partial\left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}}}{\partial R} \\ &= -a_3 R^{\alpha-1}\left(a_1 P^\alpha + a_2 W^\alpha + a_3 R^\alpha\right)^{\frac{1}{\alpha}-1}. \end{aligned}$$

Since $K^*(P,W,R) > 0$ and all terms except $-a_3$ in $K^*(P,W,R)$ are positive we require $-a_3 > 0$ or $a_3 < 0$.

Problem 67 *If we were to add another exogenous variable temperature T to the production function so that $Q = F(L,K,T)$ with $\frac{\partial F(L,K,T)}{\partial T} > 0$ (higher temperatures increase output), show that*

$$\frac{\partial \pi^*(P,W,R,T)}{\partial T} = P\frac{\partial F(L^*,K^*,T)}{\partial T} < 0.$$

Answer

We have the naive profit function:

$$\pi(L,K,P,W,R,T) = PF(L,K,T) - WL - RK.$$

From the envelope theorem:

$$\begin{aligned} \frac{\partial \pi^*(P,W,R,T)}{\partial T} &= \frac{\partial \pi(L,K,P,W,R,T)}{\partial T}|_{L=L^*,K=K^*} \\ &= P\frac{\partial F(L,K,T)}{\partial T}|_{L=L^*,K=K^*} \\ &= P\frac{\partial F(L^*,K^*,T)}{\partial T} < 0. \end{aligned}$$

Problem 68 *Using total differentials prove that*

$$\pi^*(w) = f(L^*(w)) - wL^*(w)$$

is convex given that when $Q = f(L)$ with $f'(L) > 0$ and $f''(L) < 0$.

Answer

From the first-order conditions for profit maximization we have:

$$\begin{aligned} f'(L^*) - w &= 0 \Longrightarrow f''(L^*)\,dL^* - dw = 0 \\ &\Longrightarrow \frac{dL^*}{dw} = \frac{1}{f''(L^*)} < 0. \end{aligned}$$

Now using the chain rule we have:

$$\begin{aligned} \frac{d\pi^*(w)}{dw} &= f'(L^*(w))\frac{dL^*(w)}{dw} - w\frac{dL^*(w)}{dw} - L^*(w) \\ &= \overbrace{(f'(L^*(w)) - w)}^{=0}\frac{dL^*(w)}{dw} - L^*(w) \\ &= -L^*(w) \end{aligned}$$

so that:

$$\frac{d^2\pi^*(w)}{dw^2} = -\frac{dL^*(w)}{dw} > 0$$

so that $\pi^*(w)$ is convex.

The following 2 problems are based on the following information:

Consider a firm with a production function:

$$Q = F(L, K) = L^{1/2}K^{1/3}$$

so that the naive profit function is:

$$\pi(L, K, P, W, R) = PL^{1/2}K^{1/3} - WL - RK.$$

Problem 69 *Using this information derive the firm's demand for labour $L^*(P, W, R)$, the firm's demand for capital $K^*(P, W, R)$ and the firm's supply curve $Q^*(P, W, R)$ from the first-order conditions.*

Answer

We have:

$$\begin{aligned} \frac{\partial \pi(L, K)}{\partial L} &= \frac{1}{2}PL^{-1/2}K^{1/3} - W \\ \frac{\partial \pi(L, K)}{\partial K} &= \frac{1}{3}PL^{1/2}K^{-2/3} - R \end{aligned}$$

so that the first-order conditions become:

$$\begin{aligned} \frac{\partial \pi\left(L^*, K^*\right)}{\partial L} &= \frac{1}{2} P\left(L^*\right)^{-1/2}\left(K^*\right)^{1/3}-W=0 \\ \frac{\partial \pi\left(L^*, K^*\right)}{\partial K} &= \frac{1}{3} P\left(L^*\right)^{1/2}\left(K^*\right)^{-2/3}-R=0 \end{aligned}$$

so that:

$$\begin{aligned} \left(L^*\right)^{-1/2}\left(K^*\right)^{1/3} &= 2\frac{W}{P} \Longrightarrow -\frac{1}{2}\ln\left(L^*\right)+\frac{1}{3}\ln\left(K^*\right)=\ln\left(2\frac{W}{P}\right) \\ \left(L^*\right)^{1/2}\left(K^*\right)^{-2/3} &= 3\frac{R}{P} \Longrightarrow \frac{1}{2}\ln\left(L^*\right)-\frac{2}{3}\ln\left(K^*\right)=\ln\left(3\frac{R}{P}\right). \end{aligned}$$

Combining these two results and using matrix notation we find that:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} \ln\left(L^*\right) \\ \ln\left(K^*\right) \end{bmatrix}=\begin{bmatrix} \ln\left(2\frac{W}{P}\right) \\ \ln\left(3\frac{R}{P}\right) \end{bmatrix}$$

with:

$$\det\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{bmatrix}=\frac{1}{6}.$$

Hence using Cramer's rule:

$$\begin{aligned} \ln\left(L^*\right) &= \frac{\det\begin{bmatrix} \ln\left(2\frac{W}{P}\right) & \frac{1}{3} \\ \ln\left(3\frac{R}{P}\right) & -\frac{2}{3} \end{bmatrix}}{\frac{1}{6}} = -4\ln\left(2\frac{W}{P}\right)-2\ln\left(3\frac{R}{P}\right) \\ \ln\left(K^*\right) &= \frac{\det\begin{bmatrix} -\frac{1}{2} & \ln\left(2\frac{W}{P}\right) \\ \frac{1}{2} & \ln\left(3\frac{R}{P}\right) \end{bmatrix}}{\frac{1}{6}} = -3\ln\left(2\frac{W}{P}\right)-3\ln\left(3\frac{R}{P}\right) \end{aligned}$$

so that:

$$\begin{aligned} L^*(P,W,R) &= \left(2\frac{W}{P}\right)^{-4}\left(3\frac{R}{P}\right)^{-2}=\frac{1}{144}P^6W^{-4}R^{-2} \\ K^*(P,W,R) &= \left(2\frac{W}{P}\right)^{-3}\left(3\frac{R}{P}\right)^{-3}=\frac{1}{216}P^6W^{-3}R^{-3}. \end{aligned}$$

Therefore:

$$\begin{aligned} Q^* &= \left(L^*\right)^{\frac{1}{2}}\left(K^*\right)^{\frac{1}{3}}=\left(\frac{1}{144}P^6W^{-4}R^{-2}\right)^{\frac{1}{2}}\left(\frac{1}{216}P^6W^{-3}R^{-3}\right)^{\frac{1}{3}} \\ &= \frac{1}{72}P^5W^{-3}R^{-2}. \end{aligned}$$

Problem 70 *Find the clever profit function:*

$$\pi^*(P,W,R)$$

and show that it is homogeneous of degree 1*. Now use Hotelling's lemma to find* L^**,* K^* *and* Q^* *from* $\pi^*(P,W,R)$ *and verify that you get the same answers as in the above problem.*

Answer

We have:

$$\begin{aligned}
\pi^*(P,W,R) &= PQ^*(P,W,R) - WL^*(P,W,R) - RK^*(P,W,R) \\
&= P\frac{1}{72}P^5W^{-3}R^{-2} - W\frac{1}{144}P^6W^{-4}R^{-2} - R\frac{1}{216}P^6W^{-3}R^{-3} \\
&= \left(\frac{1}{72} - \frac{1}{144} - \frac{1}{216}\right)P^6W^{-3}R^{-2} \\
&= \frac{1}{432}P^6W^{-3}R^{-2}.
\end{aligned}$$

Adding the exponents we see that $\pi^*(P,W,R)$ is homogeneous of degree $k = 6 + -3 + -2 = 1$.

We can verify Hotelling's lemma since:

$$\begin{aligned}
\frac{\partial \pi^*(P,W,R)}{\partial P} &= \frac{\partial}{\partial P}\left(\frac{1}{432}P^6W^{-3}R^{-2}\right) \\
&= \frac{6}{432}P^5W^{-3}R^{-2} \\
&= \frac{1}{72}P^5W^{-3}R^{-2} = Q^*(P,W,R) \\
\frac{\partial \pi^*(P,W,R)}{\partial W} &= \frac{\partial}{\partial W}\left(\frac{1}{432}P^6W^{-3}R^{-2}\right) \\
&= -\frac{3}{432}P^6W^{-4}R^{-2} \\
&= -\frac{1}{144}P^6W^{-4}R^{-2} = -L^*(P,W,R) \\
\frac{\partial \pi^*(P,W,R)}{\partial R} &= \frac{\partial}{\partial R}\left(\frac{1}{432}P^6W^{-3}R^{-2}\right) \\
&= -\frac{2}{432}P^6W^{-3}R^{-3} \\
&= -\frac{1}{216}P^6W^{-3}R^{-3} = -K^*(P,W,R)
\end{aligned}$$

2.2 Utility Maximization

Problem 71 *Suppose that a household maximizes utility $U(Q_1,Q_2)$ subject to the budget constraint $Y = P_1Q_1 + P_2Q_2$. The resulting clever (or indirect) utility function turns out to be:*

$$U^*(P_1,P_2,Y) = Y^{\frac{1}{3}}\left(P_1^{-\frac{1}{2}} + P_2^{-\frac{1}{2}}\right)^{\frac{2}{3}}.$$

Use this information to determine the Lagrange multiplier λ^* and the demand curve $Q_1^*(P_1,P_2,Y)$. What is the income elasticity for Q_1?

Answer

We have:

$$\begin{aligned}
\lambda^*(P_1,P_2,Y) &= \frac{\partial U^*(P_1,P_2,Y)}{\partial Y} = \frac{1}{3}Y^{-\frac{2}{3}}\left(P_1^{-\frac{1}{2}}+P_2^{-\frac{1}{2}}\right)^{\frac{2}{3}} \\
\frac{\partial U^*(P_1,P_2,Y)}{\partial P_1} &= -\frac{1}{2}\times\frac{2}{3}Y^{\frac{1}{3}}\left(P_1^{-\frac{1}{2}}+P_2^{-\frac{1}{2}}\right)^{-\frac{1}{3}}P_1^{-\frac{3}{2}}
\end{aligned}$$

so that:

$$\begin{aligned}
Q_1^*(P_1,P_2,Y) &= -\frac{\frac{\partial U^*(P_1,P_2,Y)}{\partial P_1}}{\frac{\partial U^*(P_1,P_2,Y)}{\partial Y}} \\
&= \frac{\frac{1}{3}Y^{\frac{1}{3}}\left(P_1^{-\frac{1}{2}}+P_2^{-\frac{1}{2}}\right)^{-\frac{1}{3}}P_1^{-\frac{3}{2}}}{\frac{1}{3}Y^{-\frac{2}{3}}\left(P_1^{-\frac{1}{2}}+P_2^{-\frac{1}{2}}\right)^{\frac{2}{3}}} \\
&= \frac{YP_1^{-\frac{3}{2}}}{P_1^{-\frac{1}{2}}+P_2^{-\frac{1}{2}}}.
\end{aligned}$$

The income elasticity is the exponent on Y or 1.

The following 3 problems are based on the following information:

Suppose you are running a prison. Inmates in the prison consume only two goods: Q_1 and Q_2 (say bread and water) which have prices P_1 and P_2 respectively. You as the warden, having no control over prices, wish to choose Q_1 and Q_2 so as to minimize the expenditure E on each prisoner given by:

$$E = P_1Q_1 + P_2Q_2.$$

Prison regulations stipulate that each prisoner's utility be a certain minimum amount, say U_0, so that your constraint as a warden is that Q_1 and Q_2 must satisfy:

$$U_0 = U(Q_1,Q_2) = \alpha\ln(Q_1) + (1-\alpha)\ln(Q_2).$$

Problem 72 *From the Lagrangian find the first -order conditions for the prison allocation problem and solve for the optimal values. Find*

$$E^*(P_1,P_2,U_0) = P_1Q_1^*(P_1,P_2,U_0) + P_2Q_2^*(P_1,P_2,U_0).$$

Answer

We have:

$$\mathcal{L}(\lambda,Q_1,Q_2,P_1,P_2,U_0) = P_1Q_1 + P_2Q_2 + \lambda(U_0 - \alpha\ln(Q_1) - (1-\alpha)\ln(Q_2))$$

so that the first-order conditions are:

$$\begin{aligned} U_0 - \alpha_1 \ln(Q_1^*) - \alpha_2 \ln(Q_2^*) &= 0 \Longrightarrow U_0 = \alpha \ln(Q_1^*) + (1-\alpha)\ln(Q_2^*) \\ P_1 - \lambda^* \frac{\alpha_1}{Q_1^*} &= 0 \Longrightarrow Q_1^* = \frac{\lambda^* \alpha}{P_1} \\ P_2 - \lambda^* \frac{\alpha_1}{Q_2^*} &= 0 \Longrightarrow Q_2^* = \frac{\lambda^* (1-\alpha)}{P_2}. \end{aligned}$$

Placing the second and third results in the first results in:

$$U_0 = \alpha \ln\left(\frac{\lambda^* \alpha}{P_1}\right) + (1-\alpha)\ln\left(\frac{\lambda^*(1-\alpha)}{P_2}\right)$$

so that solving for λ^* we have:

$$\lambda^* = \lambda^*(P_1, P_2, U_0) = \frac{e^{U_0} P_1^{\alpha} P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$$

so that:

$$\begin{aligned} Q_1^* &= Q_1^*(P_1, P_2, U_0) = \frac{\lambda^* \alpha}{P_1} = \frac{e^{U_0} P_1^{\alpha} P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \frac{\alpha}{P_1} \\ Q_2^* &= Q_2^*(P_1, P_2, U_0) = \frac{\lambda^*(1-\alpha)}{P_2} = \frac{e^{U_0} P_1^{\alpha} P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \frac{1-\alpha}{P_2}. \end{aligned}$$

Now:

$$\begin{aligned} E^*(P_1, P_2, U_0) &= P_1 Q_1^* + P_2 Q_2^* \\ &= P_1 \frac{e^{U_0} P_1^{\alpha} P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \frac{\alpha}{P_1} + P_2 \frac{e^{U_0} P_1^{\alpha} P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \frac{1-\alpha}{P_2} \\ &= \frac{e^{U_0} P_1^{\alpha} P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}. \end{aligned}$$

Problem 73 *Verify that:*

$$\begin{aligned} Q_1^*(P_1, P_2, U_0) &= \frac{\partial E^*(P_1, P_2, U_0)}{\partial P_1}, \\ Q_2^*(P_1, P_2, U_0) &= \frac{\partial E^*(P_1, P_2, U_0)}{\partial P_2} \\ \lambda^*(P_1, P_2, U_0) &= \frac{\partial E^*(P_1, P_2, U_0)}{\partial U_0}. \end{aligned}$$

and show that $\frac{\partial Q_1^*}{\partial P_2} = \frac{\partial Q_2^*}{\partial P1}$.

Answer

We have:

$$\begin{aligned}
\frac{\partial E^*}{\partial P_1} &= \frac{\partial}{\partial P_1}\left(\frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) = \alpha\frac{e^{U_0}P_1^{\alpha-1}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} = \frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\frac{\alpha}{P_1} = Q_1^*(P_1,P_2,U_0) \\
\frac{\partial E^*}{\partial P_2} &= \frac{\partial}{\partial P_2}\left(\frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) = (1-\alpha)\frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha-1}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} = \frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\frac{1-\alpha}{P_2} = Q_2^*(P_1,P_2,U_0) \\
\frac{\partial E^*}{\partial U_0} &= \frac{\partial}{\partial U_o}\left(\frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) = \frac{e^{U_0}P_1^{\alpha-1}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} = \lambda^*(P_1,P_2,U_0).
\end{aligned}$$

Furthermore:

$$\begin{aligned}
\frac{\partial Q_1^*}{\partial P_2} &= \frac{\partial}{\partial P_2}\left(\alpha\frac{e^{U_0}P_1^{\alpha-1}P_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) = \alpha(1-\alpha)\frac{e^{U_0}P_1^{\alpha-1}P_2^{-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \\
\frac{\partial Q_2^*}{\partial P_1} &= \frac{\partial}{\partial P_1}\left((1-\alpha)\frac{e^{U_0}P_1^{\alpha}P_2^{1-\alpha-1}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}\right) = \alpha(1-\alpha)\frac{e^{U_0}P_1^{\alpha-1}P_2^{-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}
\end{aligned}$$

and so $\frac{\partial Q_1^*}{\partial P_2} = \frac{\partial Q_2^*}{\partial P_1}$.

The following 2 problems are based on the following information:

Suppose that the household has to pay income tax on Y: t_y, and a tax on Q_1 of t_1 so that the budget constraint becomes:

$$Y \times (1-t_y) = P_1 \times (1+t_1)\,Q_1 + P_2Q_2.$$

Problem 74 *Let $U^*(P_1,P_2,Y,t_y,t_1)$ be the resulting clever utility function. Find the appropriate modification of Roy's identity that would allow you to calculate Q_1^* from $U^*(P_1,P_2,Y,t_y,t_1)$.*

Answer

The Lagrangian for this problem is:

$$\mathcal{L}(\lambda,Q_1,Q_2,P_1,P_2,Y,t_y,t_1) = U(Q_1,Q_2) + \lambda\left(Y\times(1-t_y) - P_1\times(1+t_1)\,Q_1 - P_2Q_2\right).$$

Thus:

$$\begin{aligned}
\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial Y} &= \frac{\partial}{\partial Y}\mathcal{L}(\lambda,Q_1,Q_2,P_1,P_2,Y,t_y,t_1)\,|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\
&= \lambda(1-t_y)\,|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\
&= \lambda^*(1-t_y)
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial P_1} &= \frac{\partial}{\partial P_1}\mathcal{L}(\lambda,Q_1,Q_2,P_1,P_2,Y,t_y,t_1)\,|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\ &= -\lambda(1+t_1)Q_1|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\ &= -\lambda^*(1+t_1)Q_1^*.\end{aligned}$$

Thus:

$$\frac{\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial P_1}}{\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial Y}} = \frac{-\lambda^*(1+t_1)Q_1^*}{\lambda^*(1-t_y)} = \frac{(1+t_1)Q_1^*}{(1-t_y)}$$

so that:

$$Q_1^* = -\frac{(1-t_y)}{(1+t_1)}\frac{\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial P_1}}{\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial Y}}.$$

Problem 75 *Using the envelope theorem calculate:*

$$\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial t_y} \text{ and } \frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial t_1}.$$

Use these results to show that

$$\frac{\partial U^*}{\partial t_1}\Big/\frac{\partial U^*}{\partial t_y}$$

is the budget share of good 1 *or equivalently that:*

$$Q_1^* = \frac{\frac{\partial U^*}{\partial t_1}}{\frac{\partial U^*}{\partial t_y}}\frac{Y}{P_1}.$$

Answer

We have:

$$\begin{aligned}\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial t_y} &= \frac{\partial}{\partial t_y}\mathcal{L}(\lambda,Q_1,Q_2,P_1,P_2,Y,t_y,t_1)\,|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\ &= -\lambda Y|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\ &= -\lambda^* Y\end{aligned}$$

and

$$\begin{aligned}\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial t_1} &= \frac{\partial}{\partial t_1}\mathcal{L}(\lambda,Q_1,Q_2,P_1,P_2,Y,t_y,t_1)\,|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\ &= -\lambda P_1 Q_1|_{\lambda=\lambda^*,Q_1=Q_1^*,Q_2=Q_2^*} \\ &= -\lambda^* P_1 Q_1^*.\end{aligned}$$

Thus:

$$\frac{\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial t_1}}{\frac{\partial U^*(P_1,P_2,Y,t_y,t_1)}{\partial t_y}} = \frac{-\lambda^* P_1 Q_1^*}{-\lambda^* Y} = \frac{P_1 Q_1^*}{Y} = \text{ budget share of } Q_1.$$

or:

$$Q_1^* = \frac{\frac{\partial U^*}{\partial t_1}}{\frac{\partial U^*}{\partial t_y}} \frac{Y}{P_1}.$$

The following 3 problems are based on the following information:

Consider the household's problem of maximizing utility as a function of three goods: $U(Q_1, Q_2, Q_3)$ subject to the budget constraint:

$$Y = P_1 Q_1 + P_2 Q_2 + P_3 Q_3.$$

Problem 76 *Set up the Lagrangian for this problem and find the first-order conditions. Show from the first-order conditions that:*

$$\frac{MU_1}{P_1} = \frac{MU_2}{P_2} = \frac{MU_3}{P_3} = \lambda^*$$

Answer

We have:

$$\mathcal{L} = U(Q_1, Q_2, Q_3) + \lambda (Y - P_1 Q_1 - P_2 Q_2 - P_3 Q_3)$$

so that the first-order conditions are:

$$\begin{aligned} Y - P_1 Q_1^* - P_2 Q_2^* - P_3 Q_3^* &= 0 \\ \frac{\partial U(Q_1^*, Q_2^*, Q_3^*)}{\partial Q_1} - \lambda^* P_1 &= 0 \\ \frac{\partial U(Q_1^*, Q_2^*, Q_3^*)}{\partial Q_2} - \lambda^* P_2 &= 0 \\ \frac{\partial U(Q_1^*, Q_2^*, Q_3^*)}{\partial Q_3} - \lambda^* P_3 &= 0. \end{aligned}$$

Since $MU_i = \frac{\partial U(Q_1^*, Q_2^*, Q_3^*)}{\partial Q_i}$ we have the desired result.

Problem 77 *Using the envelope theorem, prove that λ^* is the marginal utility of income. Prove Roy's identity; that is how to calculate $Q_1^*(P_1, P_2, P_3, Y)$, from the clever utility function: $U^*(P_1, P_2, P_3, Y)$.*

Answer

We have:

$$\begin{aligned}\frac{\partial U^*(P_1,P_2,P_3,Y)}{\partial Y} &= \frac{\partial \mathcal{L}(\lambda,Q_1,Q_2,Q_3,P_1,P_2,P_3,Y)}{\partial Y}|_{Q_i=Q_i^*,\lambda=\lambda^*} \\ &= \lambda|_{Q_i=Q_i^*,\lambda=\lambda^*} \\ &= \lambda^*\end{aligned}$$

and:

$$\begin{aligned}\frac{\partial U^*(P_1,P_2,P_3,Y)}{\partial P_1} &= \frac{\partial \mathcal{L}(\lambda,Q_1,Q_2,Q_3,P_1,P_2,P_3,Y)}{\partial P_1}|_{Q_i=Q_i^*,\lambda=\lambda^*} \\ &= -\lambda Q_1|_{Q_i=Q_i^*,\lambda=\lambda^*} \\ &= -\lambda^* Q_1^*\end{aligned}$$

so that:

$$-\frac{\frac{\partial U^*(P_1,P_2,P_3,Y)}{\partial P_1}}{\frac{\partial U^*(P_1,P_2,P_3,Y)}{\partial Y}} = -\frac{-\lambda^* Q_1^*}{\lambda^*} = Q_1^*.$$

Problem 78 *Consider making this problem an unconstrained maximization problem by using the budget constraint to solve out Q_3 as:*

$$Q_3 = \frac{Y - P_1Q_1 - P_2Q_2}{P_3}.$$

Thus maximizing $U(Q_1,Q_2,Q_3)$ subject to the income constraint is equivalent to the unconstrained problem of maximizing:

$$V(Q_1,Q_2,P_1,P_2,P_3,Y)$$

over Q_1 and Q_2 where:

$$V(Q_1,Q_2,P_1,P_2,P_3,Y) = U\left(Q_1,Q_2,\frac{Y - P_1Q_1 - P_2Q_2}{P_3}\right).$$

Let

$$V^*(P_1,P_2,P_3,Y) = V(Q_1^*,Q_2^*,P_1,P_2,P_3,Y)$$

be the clever version of V. Use the envelope theorem to show how to calculate $Q_1^*(P_1,P_2,P_3,Y)$ from $V^*(P_1,P_2,P_3,Y)$.

Answer

We have:

$$\begin{aligned}
\frac{\partial V^*(P_1,P_2,P_3,Y)}{\partial Y} &= \frac{\partial U\left(Q_1,Q_2,\frac{Y-P_1Q_1-P_2Q_2}{P_3}\right)}{\partial Y}|_{Q_1=Q_1^*,Q_2=Q_2^*} \\
&= -\frac{1}{P_3}\frac{\partial U\left(Q_1,Q_2,\frac{Y-P_1Q_1-P_2Q_2}{P_3}\right)}{\partial Q_3}|_{Q_1=Q_1^*,Q_2=Q_2^*} \\
&= -\frac{1}{P_3}\frac{\partial U\left(Q_1^*,Q_2^*,\frac{Y-P_1Q_1^*-P_2Q_2^*}{P_3}\right)}{\partial Q_3} \\
&= -\frac{1}{P_3}\frac{\partial U(Q_1^*,Q_2^*,Q_3^*)}{\partial Q_3} = -\frac{MU_3}{P_3}
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\partial V^*(P_1,P_2,P_3,Y)}{\partial P_1} &= \frac{\partial U\left(Q_1,Q_2,\frac{Y-P_1Q_1-P_2Q_2}{P_3}\right)}{\partial P_1}|_{Q_1=Q_1^*,Q_2=Q_2^*} \\
&= -\frac{Q_1}{P_3}\frac{\partial U\left(Q_1,Q_2,\frac{Y-P_1Q_1-P_2Q_2}{P_3}\right)}{\partial Q_3}|_{Q_1=Q_1^*,Q_2=Q_2^*} \\
&= -\frac{Q_1^*}{P_3}\frac{\partial U\left(Q_1^*,Q_2^*,\frac{Y-P_1Q_1^*-P_2Q_2^*}{P_3}\right)}{\partial Q_3} \\
&= -\frac{Q_1^*}{P_3}\frac{\partial U(Q_1^*,Q_2^*,Q_3^*)}{\partial Q_3} = -\frac{Q_1^*}{P_3}MU_3.
\end{aligned}$$

Therefore:

$$-\frac{\frac{\partial V^*(P_1,P_2,P_3,Y)}{\partial P_1}}{\frac{\partial V^*(P_1,P_2,P_3,Y)}{\partial Y}} = \frac{-\frac{Q_1^*}{P_3}MU_3}{-\frac{MU_3}{P_3}} = Q_1^*.$$

Problem 79 *Consider a Cobb-Douglas utility function with n goods:*

$$U = \alpha_1 \ln(Q_1) + \alpha_2 \ln(Q_2) + \cdots + \alpha_n \ln(Q_n)$$

where: $\alpha_i \geq 0$*,*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

and the budget constraint is:

$$Y = P_1Q_1 + P_2Q_2 + \cdots + P_nQ_n.$$

Find the utility maximizing values of Q_i^* *and use this to construct the indirect utility function* $U^*(P_1,P_2\ldots,P_n,Y)$*. Verify Roy's identify:*

$$Q_i^* = -\frac{\frac{\partial U^*(P_1,P_2\ldots,P_n,Y)}{\partial P_i}}{\frac{\partial U^*(P_1,P_2\ldots,P_n,Y)}{\partial Y}}.$$

Verify that $U^*(P_1,P_2\ldots,P_n,Y)$ *is homogeneous of degree* 0*.*

Answer

The Lagrangian is:

$$\begin{aligned}\mathcal{L}(\lambda, Q_1, Q_2, \dots Q_n) &= \alpha_1 \ln(Q_1) + \alpha_2 \ln(Q_2) + \cdots + \alpha_n \ln(Q_n) \\ &\quad + \lambda (Y - P_1 Q_1 - P_2 Q_2 - \cdots - P_n Q_n)\end{aligned}$$

so that the first-order conditions are:

$$\begin{aligned}\frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*, \dots Q_n^*)}{\partial \lambda} &= 0 = Y - P_1 Q_1^* - P_2 Q_2^* - \cdots - P_n Q_n^* \\ \frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*, \dots Q_n^*)}{\partial Q_1} &= 0 = \frac{\alpha_1}{Q_1^*} - \lambda^* P_1 \Longrightarrow \alpha_1 = \lambda^* P_1 Q_1^* \\ \frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*, \dots Q_n^*)}{\partial Q_2} &= 0 = \frac{\alpha_2}{Q_2^*} - \lambda^* P_2 \Longrightarrow \alpha_2 = \lambda^* P_2 Q_2^* \\ &\vdots \\ \frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*, \dots Q_n^*)}{\partial Q_n} &= 0 = \frac{\alpha_n}{Q_n^*} - \lambda^* P_n \Longrightarrow \alpha_n = \lambda^* P_n Q_n^*.\end{aligned}$$

Adding up the last n first-order conditions we have:

$$\begin{aligned}\alpha_1 + \alpha_2 + \cdots + \alpha_n &= \lambda^* P_1 Q_1^* + \lambda^* P_2 Q_2^* + \cdots + \lambda^* P_n Q_n^* \\ &\Longrightarrow \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_n}_{=1} = \lambda^* \underbrace{(P_1 Q_1^* + P_2 Q_2^* + \cdots + P_n Q_n^*)}_{=Y} \\ &\Longrightarrow \lambda^* = \frac{1}{Y}.\end{aligned}$$

It then follows that:

$$\begin{aligned}\alpha_i &= \lambda^* P_i Q_i^* \\ &= \frac{P_i Q_i^*}{Y} \\ &\Longrightarrow Q_i^* = \frac{\alpha_i Y}{P_i}.\end{aligned}$$

We therefore have:

$$\begin{aligned}U^*(P_1, P_2 \dots, P_n, Y) &= \alpha_1 \ln(Q_1^*) + \alpha_2 \ln(Q_2^*) + \cdots + \alpha_n \ln(Q_n^*) \\ &= \alpha_1 \ln\left(\frac{\alpha_1 Y}{P_1}\right) + \alpha_2 \ln\left(\frac{\alpha_2 Y}{P_2}\right) + \cdots + \alpha_n \ln\left(\frac{\alpha_n Y}{P_n}\right) \\ &= \alpha_1 (\ln(\alpha_1) + \ln(Y) - \ln(P_1)) + \cdots + \alpha_n (\ln(\alpha_n) + \ln(Y) - \ln(P_n)) \\ &= \ln(Y) - \alpha_1 \ln(P_1) + \alpha_2 \ln(P_2) + \cdots + \alpha_n \ln(P_n) - c\end{aligned}$$

where the coefficient on $\ln(Y)$ is one since the $\alpha_i\,'s$ sum to 1 and the constant c is:

$$c = \alpha_1 \ln(\alpha_1) + \alpha_2 \ln(\alpha_2) + \cdots + \alpha_n \ln(\alpha_n).$$

We therefore have:

$$\begin{aligned} \frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial Y} &= \frac{1}{Y} = \lambda^* \\ \frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial P_i} &= -\frac{\alpha_i}{P_i} \end{aligned}$$

so that:

$$-\frac{\frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial P_i}}{\frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial Y}} = -\frac{-\frac{\alpha_i}{P_i}}{\frac{1}{Y}} = \frac{\alpha_i Y}{P_i} = Q_i^*.$$

The clever utility function is homogeneous of degree 0 since:

$$\begin{aligned} U^*(\lambda P_1, \lambda P_2 \ldots, \lambda P_n, \lambda Y) &= \ln(\lambda Y) - \alpha_1 \ln(\lambda P_1) + \alpha_2 \ln(\lambda P_2) + \cdots + \alpha_n \ln(\lambda P_n) - c \\ &= \ln(\lambda) \left(1 - \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}_{=1}\right) \\ &\quad + \ln(Y) - \alpha_1 \ln(P_1) + \alpha_2 \ln(P_2) + \cdots + \alpha_n \ln(P_n) - c \\ &= \underbrace{\lambda^0}_{=1} (\ln(Y) - \alpha_1 \ln(P_1) + \alpha_2 \ln(P_2) + \cdots + \alpha_n \ln(P_n) - c) \\ &= \lambda^0 U^*(P_1, P_2 \ldots, P_n, Y) \end{aligned}$$

and so $U^*(P_1, P_2 \ldots, P_n, Y)$ is homogeneous of degree 0.

Problem 80 *The CES utility function (**C**onstant **E**lasticity of **S**ubstitution) is given by:*

$$U = \sum_{i=1}^{n} \alpha_i \left(\frac{Q_i^{1+\rho} - 1}{1+\rho} \right)$$

where $\rho < 0$. This includes the Cobb-Douglas as a special case where $\rho = -1$ since from L'Hôpital's rule it follows that:

$$\lim_{\rho \to -1} \frac{Q_i^{1+\rho} - 1}{1+\rho} = \ln(Q_i).$$

Calculate the demand functions Q_i^* for the CES utility function and the indirect utility function $U^*(P_1, P_2 \ldots, P_n, Y)$. Verify Roy's identity. Verify that $U^*(P_1, P_2 \ldots, P_n, Y)$ is homogeneous of degree 0. (Note: in this problem we use the summation notation, covered in the next chapter, to make things more compact. If you are uncomfortable with using this notation write everything as a conventional sum, for example:

$$\begin{aligned} U &= \sum_{i=1}^{n} \alpha_i \left(\frac{Q_i^{1+\rho} - 1}{1+\rho} \right) \\ &= \alpha_1 \left(\frac{Q_1^{1+\rho} - 1}{1+\rho} \right) + \alpha_2 \left(\frac{Q_2^{1+\rho} - 1}{1+\rho} \right) + \cdots + \alpha_n \left(\frac{Q_n^{1+\rho} - 1}{1+\rho} \right). \end{aligned}$$

Answer

The Lagrangian is:

$$\begin{aligned}\mathcal{L}(\lambda, Q_1, Q_2, \ldots Q_n) &= \sum_{i=1}^{n} \alpha_i \left(\frac{Q_i^{1+\rho} - 1}{1+\rho} \right) \\ &\quad + \lambda \left(Y - \sum_{i=1}^{n} P_i Q_i \right)\end{aligned}$$

so that the first-order conditions are:

$$\begin{aligned}\frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*, \ldots Q_n^*)}{\partial \lambda} &= 0 = Y - \sum_{i=1}^{n} P_i Q_i^* = 0 \Longrightarrow Y = \sum_{i=1}^{n} P_i Q_i^* \\ \frac{\partial \mathcal{L}(\lambda^*, Q_1^*, Q_2^*, \ldots Q_n^*)}{\partial Q_i} &= 0 = \alpha_1 (Q_i^*)^{\rho} - \lambda^* P_i \Longrightarrow P_i Q_i^* = (\lambda^*)^{\frac{1}{\rho}} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \\ \text{for } i &= 1, 2, \ldots n.\end{aligned}$$

Adding up the implications of the last n first-order conditions we have:

$$\begin{aligned}Y &= \sum_{i=1}^{n} P_i Q_i^* \\ &= \sum_{i=1}^{n} (\lambda^*)^{\frac{1}{\rho}} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \\ &= (\lambda^*)^{\frac{1}{\rho}} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right) \\ \Longrightarrow & \ (\lambda^*)^{\frac{1}{\rho}} = \frac{Y}{\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1}}.\end{aligned}$$

Once we have the Lagrange multiplier we can calculate the demand curves using:

$$\begin{aligned}\alpha_i (Q_i^*)^{\rho} - \lambda^* P_i &= 0 \Longrightarrow \\ Q_i^* &= (\lambda^*)^{\frac{1}{\rho}} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}} \\ &= \frac{\alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}} Y}{\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1}}.\end{aligned}$$

Now:

$$
\begin{aligned}
U^*(P_1, P_2 \ldots, P_n, Y) &= \sum_{i=1}^{n} \alpha_i \left(\frac{Q_i^{*1+\rho} - 1}{1+\rho} \right) \\
&= \frac{1}{1+\rho} \sum_{i=1}^{n} \alpha_i Q_i^{*1+\rho} - \frac{\sum_{i=1}^{n} \alpha_i}{1+\rho} \\
&= \frac{1}{1+\rho} \sum_{i=1}^{n} \alpha_i \left(\frac{\alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}} Y}{\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1}} \right)^{1+\rho} - \frac{\sum_{i=1}^{n} \alpha_i}{1+\rho} \\
&= \frac{Y^{1+\rho}}{1+\rho} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right)^{-\rho} - \frac{\sum_{i=1}^{n} \alpha_i}{1+\rho}.
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial Y} &= Y^{\rho} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right)^{-\rho} \\
\frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial P_i} &= \frac{-\rho Y^{1+\rho}}{1+\rho} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right)^{-\rho-1} \times \alpha_i^{-\frac{1}{\rho}} \left(\frac{1}{\rho} + 1 \right) P_i^{\frac{1}{\rho}} \\
&= -Y^{1+\rho} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right)^{-(1+\rho)} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}}
\end{aligned}
$$

so that:

$$
\begin{aligned}
-\frac{\frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial P_i}}{\frac{\partial U^*(P_1, P_2 \ldots, P_n, Y)}{\partial Y}} &= -\frac{-Y^{1+\rho} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right)^{-(1+\rho)} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}}}{Y^{\rho} \left(\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1} \right)^{-\rho}} \\
&= \frac{\alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}} Y}{\sum_{i=1}^{n} \alpha_i^{-\frac{1}{\rho}} P_i^{\frac{1}{\rho}+1}} = Q_i^*.
\end{aligned}
$$

2.3 Mark Maximization

The following 7 problems are based on the following information:

Consider the following abstract and irrelevant problem: a student has to write an exam with n questions. He has T minutes to write the exam. Let t_i be the number of minutes he spends on question i. Suppose the marks he gets for question i is determined by the function: $M_i(t_i)$ where:

$$
\begin{aligned}
M_i'(t_i) &> 0 \\
M_i''(t_i) &< 0.
\end{aligned}
$$

The Lagrangian for this problem is then given by:

$$\mathcal{L}(\lambda, t_1, t_2, \ldots t_n, T) = \sum_{i=1}^{n} M_i(t_i) + \lambda\left(T - \sum_{i=1}^{n} t_i\right)$$

Problem 81 *What in every day language is the objective and constraint in this problem? What are the first order conditions for this problem? Using economic terminology, what do these first order conditions express?*

Answer

The first-order conditions are:

$$\begin{aligned} T - \sum_{i=1}^{n} t_i^* &= 0 \\ M_i'(t_i^*) - \lambda^* &= 0 \text{ for } i = 1, 2, \ldots n. \end{aligned}$$

The first conditions states that the optimal solution obeys the time constraint while the latter conditions state that

$$M_i'(t_i^*) = \lambda^* \text{for } i = 1, 2, \ldots n$$

or that the marginal return on each question must be the same.

Problem 82 *Will a solution to the first-order conditions be a global maximum? What are the second order conditions using the bordered Hessian for this problem for the case where $n = 2$ and show that these second order conditions will be satisfied.*

Answer

Since $\lambda^* = M_i'(t_i^*) > 0$ and since the objective function is concave while the constraint function is linear, it follows that a solution to the first-order conditions will be a global maximum.

We have:

$$\det[H] = \det\begin{bmatrix} 0 & -1 & -1 \\ -1 & M_1''(t_1^*) & 0 \\ -1 & 0 & M_2''(t_2^*) \end{bmatrix} = -M_1''(t_1^*) - M_2''(t_2^*) > 0.$$

Problem 83 *From the first order conditions show that $\lambda^*(T) > 0$. Let $t_i^*(T)$ be the value of t_i which solves the maximization problem. Consider the function:*

$$M^*(T) = \sum_{i=1}^{n} M_i(t_i^*(T))$$

Use the envelope theorem to show that:

$$\frac{\partial M^*(T)}{\partial T} = \lambda^*(T).$$

What then is the appropriate interpretation of $\lambda^*(T)$?

Answer

Since $M_i'(t) > 0$ then $\lambda^*(T) = M_i'(t_i^*) > 0$. Using the envelope theorem we have:

$$\begin{aligned}\frac{\partial M^*(T)}{\partial T} &= \frac{\partial}{\partial T}\mathcal{L}(\lambda, t_1, t_2, \dots t_n, T)\,|_{\lambda=\lambda^*, t_i=t_i^*} \\ &= \lambda|_{\lambda=\lambda^*, t_i=t_i^*} \\ &= \lambda^*(T).\end{aligned}$$

Thus $\lambda^*(T)$ is the marginal mark.

Problem 84 *For the case where $n = 2$ and using the total differential of the first order conditions, show that there is a diminishing marginal return to extra time $\left(\frac{\partial \lambda^*}{\partial T} < 0\right)$ and that an increase in T always leads to an increase in the amount of time spend on each question*

Answer

For $n = 2$ we have the total differential:

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & M_1''(t_1^*) & 0 \\ -1 & 0 & M_2''(t_2^*) \end{bmatrix} \begin{bmatrix} d\lambda^* \\ dt_1^* \\ dt_2^* \end{bmatrix} + \begin{bmatrix} dT \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where as usual the square matrix is the Hessian and has a positive determinant since:

$$\det[H] = \det \begin{bmatrix} 0 & -1 & -1 \\ -1 & M_1''(t_1^*) & 0 \\ -1 & 0 & M_2''(t_2^*) \end{bmatrix} = -M_1''(t_1^*) - M_2''(t_2^*) > 0.$$

It follows then that:

$$\begin{bmatrix} 0 & -1 & -1 \\ -1 & M_1''(t_1^*) & 0 \\ -1 & 0 & M_2''(t_2^*) \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda^*}{\partial T} \\ \frac{\partial t_1^*}{\partial T} \\ \frac{\partial t_2^*}{\partial T} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

so that using Cramer's rule:

$$\frac{\partial\lambda^*}{\partial T} = \frac{\det\begin{bmatrix} -1 & -1 & -1 \\ 0 & M_1''(t_1^*) & 0 \\ 0 & 0 & M_2''(t_2^*) \end{bmatrix}}{\det[H]} = -\frac{M_1''(t_1^*)\,M_2''(t_2^*)}{\det[H]} < 0$$

$$\frac{\partial t_1^*}{\partial T} = \frac{\det\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & M_2''(t_2^*) \end{bmatrix}}{\det[H]} = -\frac{M_2''(t_2^*)}{\det[H]} > 0$$

$$\frac{\partial t_2^*}{\partial T} = \frac{\det\begin{bmatrix} 0 & -1 & -1 \\ -1 & M_1''(t_1^*) & 0 \\ -1 & 0 & 0 \end{bmatrix}}{\det[H]} = -\frac{M_1''(t_1^*)}{\det[H]} > 0.$$

Problem 85 *Now prove the same result for the general case by showing that $M^*(T)$ is a concave function of T.*

Answer

From the constraint we have:

$$T = \sum_{i=1}^{n} t_i^*(T) = 0 \Longrightarrow 1 = \sum_{i=1}^{n} \frac{\partial t_i^*(T)}{\partial T}$$

so that at least one $\frac{\partial t_i^*(T)}{\partial T}$ must be positive. For that i we have:

$$M_i'(t_i^*(T)) = \lambda^*(T) \Longrightarrow \underbrace{M_i''(t_i^*(T))}_{-}\underbrace{\frac{\partial t_i^*(T)}{\partial T}}_{+} = \frac{\partial\lambda^*(T)}{\partial T}$$

$$\Longrightarrow \frac{\partial\lambda^*(T)}{\partial T} < 0.$$

It follows then that $M^*(T)$ is a concave function of T since from the envelope theorem: $\lambda^*(T) = \frac{\partial M^*(T)}{\partial T}$ so that:

$$\frac{\partial\lambda^*(T)}{\partial T} = \frac{\partial^2 M^*(T)}{\partial T^2} < 0.$$

From the first-order conditions and differentiating with respect to T we have now for **any** i:

$$M_i(t_i^*(T)) = \lambda^*(T) \Longrightarrow M_i''(t_i^*(T))\frac{\partial t_i^*(T)}{\partial T} = \frac{\partial\lambda^*(T)}{\partial T}$$

$$\Longrightarrow \frac{\partial t_i^*(T)}{\partial T} = \frac{\frac{\partial\lambda^*(T)}{\partial T}}{M_i''(t_i^*(T))} > 0$$

since $M_i''(t_i^*(T)) < 0$ and $\frac{\partial\lambda^*(T)}{\partial T} < 0$ from above.

Problem 86 *Suppose that the function* $M_i(t_i)$ *is given by:*

$$M_i(t_i) = m \times \left(1 - e^{-\delta t_i}\right), \ \delta > 0.$$

Show that m *is the maximum obtainable mark on question* i *, that this form satisfies:* $M_i'(t_i) > 0$, $M_i''(t_i) < 0$, *and that better students will have a higher value of* δ.

Answer

Since $e^{-\delta t_i} \to 0$ as $t_i \to \infty$ and $e^{-\delta t_i} > 0$ we have:

$$M_i(t_i) < m \text{ and } M_i(t_i) \to m \text{ as } t_i \to \infty.$$

Furthermore:

$$M_i'(t_i) = \delta m e^{-\delta t_i} > 0, M_i''(t_i) = -\delta^2 m e^{-\delta t_i} < 0$$

and

$$\frac{\partial M_i(t_i)}{\partial \delta} = t_i m e^{-\delta t_i} > 0$$

so the higher δ is the higher the grade for a given amount of time t_i.

Problem 87 *Using the information in the previous question show that it is optimal to spend an equal amount of time on each question (i.e. that:* $t_i^*(T) = \frac{T}{n}$*) and:*

$$\begin{aligned} \lambda^*(T) &= \delta \times m \times e^{-\frac{\delta}{n}T} \\ M^*(T) &= n \times m \times (1 - e^{-\frac{\delta}{n}T}). \end{aligned}$$

If $T = 60$ *and* $n = 3$, *what must* δ *be in order for the student to receive a 70% on the exam?*

Answer

From the first-order conditions:

$$M_i'(t_i^*) = \lambda^* \Longrightarrow \delta m e^{-\delta t_i^*} = \lambda^* \Longrightarrow t_i^* = -\frac{\ln\left(\frac{\lambda^*}{\delta m}\right)}{\delta}$$

so that:

$$\begin{aligned} T - \sum_{i=1}^{n} t_i^* &= 0 \Longrightarrow -\frac{n \ln\left(\frac{\lambda^*}{\delta m}\right)}{\delta} = T \Longrightarrow \lambda^*(T) = \delta \times m \times e^{-\frac{\delta}{n}T} \\ &\Longrightarrow \ t_i^*(T) = -\frac{\ln\left(\frac{\delta \times m \times e^{-\frac{\delta}{n}T}}{\delta m}\right)}{\delta} = \frac{T}{n} \\ &\Longrightarrow \ M^*(T) = \sum_{i=1}^{n} m \times \left(1 - e^{-\delta t_i}\right) = n \times m \times (1 - e^{-\frac{\delta}{n}T}). \end{aligned}$$

If $T = 60$ and $n = 3$, then:

$$\begin{aligned} M^*(60) &= 3 \times m \times (1 - e^{-\frac{\delta}{3}60}) = 3 \times m \times 0.7 \\ &\Longrightarrow 1 - e^{-\frac{\delta}{3}60} = 0.7 \\ &= e^{-\frac{\delta}{3}60} = 0.3 \\ &\Longrightarrow -\delta \times 20 = \ln(0.3) \\ &\Longrightarrow \delta = -\frac{1}{20}\ln(0.3) = 0.0602. \end{aligned}$$

The following 5 problems are based on the following information:

Consider an airline having a total of Q seats to offer the travelling public. Let Q_1 be the number of seats devoted to business class travellers and let Q_2 be the number of seats offered to tourist class travellers. The revenue from business travellers is $R_1(Q_1)$ with marginal revenue: $MR_1(Q_1) = R_1'(Q_1) > 0$ and $R_1''(Q_1) < 0$, and the revenue from tourist class travellers is $R_2(Q_2)$ with $MR_2(Q_2) = R_2'(Q_2) > 0$ and $R_2''(Q_2) < 0$. The airline wishes to allocate the given Q seats, which is equal to $Q_1 + Q_2$, so as to maximize the total revenue it gets:

$$R(Q_1, Q_2) = R_1(Q_1) + R_2(Q_2).$$

Problem 88 *Set up the Lagrangian for this problem and find the first order conditions. Show that the firm will choose Q_1^* and Q_2^* so as to equalize the marginal revenue from business and tourist class travellers.*

Answer

To maximizing profits the airline must maximize the revenue from selling Q seats. Consider then maximizing $R(Q_1, Q_2)$ subject to the constraint $Q = Q_1 + Q_2$ which leads to the Lagrangian:

$$\mathcal{L}(\lambda, Q_1, Q_2, Q) = R_1(Q_1) + R_2(Q_2) + \lambda(Q - Q_1 - Q_2)$$

so that the first-order conditions are:

$$\begin{aligned} Q - Q_1^* - Q_2^* &= 0 \\ R_1'(Q_1^*) - \lambda^* &= 0 \\ R_2'(Q_2^*) - \lambda^* &= 0. \end{aligned}$$

It follows then that:

$$MR_1(Q_1^*) = MR_2(Q_2^*) = \lambda^*$$

so that the firm equates the marginal revenue from the two markets.

Problem 89 *The airline's revenue as a function of Q will be given by:*

$$R^*(Q) = R_1(Q_1^*(Q)) + R_2(Q_2^*(Q)).$$

Use the envelope theorem to find $\frac{\partial R^*(Q)}{\partial Q}$ and show that marginal revenue in the two markets will be equal to $\frac{\partial R^*(Q)}{\partial Q}$.

Answer

We have:

$$\begin{aligned}\frac{\partial R^*(Q)}{\partial Q} &= \frac{\partial \mathcal{L}(\lambda, Q_1, Q_2, Q)}{\partial Q}|_{Q_1=Q_1^*, Q_2=Q_2^*, \lambda=\lambda^*} \\ &= \lambda|_{Q_1=Q_1^*, Q_2=Q_2^*, \lambda=\lambda^*} \\ &= \lambda^*.\end{aligned}$$

From the first-order conditions we then have:

$$\frac{\partial R^*(Q)}{\partial Q} = \lambda^* = R_1'(Q_1^*) = R_2'(Q_2^*).$$

Problem 90 *Revenue in the two markets is given by:*

$$R_1(Q_1) = P_1(Q_1)Q_1, \; R_2(Q_2) = P_2(Q_2)Q_2$$

where $P_1(Q_1)$ and $P_2(Q_2)$ are the inverse demand functions satisfying $P_1'(Q_1) < 0$ and $P_2'(Q_2) < 0$. Show that it is the market with the least elastic inverse demand curve that will have the higher price.

Answer

We have:

$$\begin{aligned}R_1'(Q_1) &= P_1(Q_1) + P_1'(Q_1)Q_1 = P_1(Q_1)(1 + \eta_1(Q_1)) \\ R_2'(Q_2) &= P_2(Q_2) + P_2'(Q_2)Q_2 = P_2(Q_2)(1 + \eta_2(Q_2))\end{aligned}$$

where the elasticities of the inverse demand curves are:

$$\eta_1(Q_1) = \frac{P_1'(Q_1)Q_1}{P_1(Q_1)}, \; \eta_2(Q_2) = \frac{P_2'(Q_2)Q_2}{P_2(Q_2)}.$$

Since: $R_1'(Q_1^*) = R_2'(Q_2^*)$ we have:

$$P_1(Q_1^*)(1 + \eta_1(Q_1^*)) = P_2(Q_2^*)(1 + \eta_2(Q_2^*))$$

we have:

$$\frac{P_1(Q_1^*)}{P_2(Q_2^*)} = \frac{(1 + \eta_2(Q_2^*))}{(1 + \eta_1(Q_1^*))}$$

so that:

$$\eta_2(Q_2^*) > \eta_1(Q_1^*) \Longrightarrow P_1(Q_1^*) > P_2(Q_2^*).$$

Problem 91 *Comment on the relationship between mark maximization and this problem.*

Answer

2.4 Cost Minimization

Problem 92 *Suppose we are given a function:*

$$y = f(x_1, x_2, \ldots x_n)$$

where the i^{th} elasticity is:

$$\eta_i = \frac{\partial y}{\partial x_i}\frac{x_i}{y} = \frac{\partial f(x_1, x_2, \ldots x_n)}{\partial x_i}\frac{x_i}{f(x_1, x_2, \ldots x_n)}.$$

We can rewrite the function as:

$$\tilde{y} = \ln\left(f\left(e^{\tilde{x}_1}, e^{\tilde{x}_2}, \ldots e^{\tilde{x}_n}\right)\right)$$

where $\tilde{y} = \ln(y)$ and $\tilde{x}_i = \ln(x_i)$. Show that:

$$\eta_i = \frac{\partial \tilde{y}}{\partial \tilde{x}_i} = \frac{\partial \ln(y)}{\partial \ln(x_i)}$$

Answer

We have:

$$\begin{aligned}
\frac{\partial \tilde{y}}{\partial \tilde{x}_i} &= \frac{\partial}{\partial \tilde{x}_i}\ln\left(f\left(e^{\tilde{x}_1}, e^{\tilde{x}_2}, \ldots e^{\tilde{x}_n}\right)\right) \\
&= \frac{1}{f\left(e^{\tilde{x}_1}, e^{\tilde{x}_n}, \ldots e^{\tilde{x}_n}\right)}\frac{\partial f\left(e^{\tilde{x}_1}, e^{\tilde{x}_2}, \ldots e^{\tilde{x}_n}\right)}{\partial x_i}\frac{\partial e^{\tilde{x}_i}}{\partial x_i} \\
&= \frac{1}{f(x_1, x_2, \ldots x_n)}\frac{\partial f(x_1, x_2, \ldots x_n)}{\partial x_i}e^{\tilde{x}_i} \\
&= \frac{1}{f(x_1, x_2, \ldots x_n)}\frac{\partial f(x_1, x_2, \ldots x_n)}{\partial x_i}x_i = \eta_i.
\end{aligned}$$

Problem 93 *Given the firm's cost function: $C^*(Q, W, R)$ what are the elasticities with respect to Q, W and R?*

Answer

We have the elasticity with respect to Q of:

$$\frac{\partial \ln(C^*(Q, W, R))}{\partial \ln(Q)} = \frac{\partial C^*(Q, W, R)}{\partial Q}\frac{Q}{C^*(Q, W, R)} = \frac{\text{Marginal Cost}}{\text{Average Cost}}.$$

Note that if there are increasing returns to scale then marginal cost is less than average cost and the elasticity will be less than 1. If there are constant returns to scale marginal cost equals average cost so the elasticity will be 1 while if there are decreasing returns to scale then marginal cost is greater than average cost and the elasticity will be greater than 1.

We have the elasticity with respect to W of:

$$\begin{aligned}\frac{\partial \ln \left(C^*\left(Q,W,R\right)\right)}{\partial \ln \left(W\right)} &= \frac{\partial C^*\left(Q,W,R\right)}{\partial W}\frac{W}{C^*\left(Q,W,R\right)} \\ &= \frac{W \times L^*\left(Q,W,R\right)}{C^*\left(Q,W,R\right)} \\ &= \text{labour's cost share.}\end{aligned}$$

We have the elasticity with respect to R of:

$$\begin{aligned}\frac{\partial \ln \left(C^*\left(Q,W,R\right)\right)}{\partial \ln \left(R\right)} &= \frac{\partial C^*\left(Q,W,R\right)}{\partial R}\frac{R}{C^*\left(Q,W,R\right)} \\ &= \frac{R \times K^*\left(Q,W,R\right)}{C^*\left(Q,W,R\right)} \\ &= \text{capital's cost share.}\end{aligned}$$

Problem 94 *Suppose the firm's cost function is given by:*

$$C^*\left(Q,W,R\right) = Q^{\frac{1}{3}}W^{\frac{1}{4}}R^{\delta}.$$

What is the exponent on R (i.e., δ) equal to? What do the exponents on Q and W tell you? Calculate the conditional factor demands: L^* and K^* as well as λ^*.

Answer

Since $C^*\left(Q,W,R\right)$ is homogeneous of degree 1 in W and R the exponents on W and R must sum to 1 here or $\delta = 1 - \frac{1}{4} = \frac{3}{4}$. Since:

$$\frac{\partial \ln \left(C^*\left(Q,W,R\right)\right)}{\partial \ln \left(Q\right)} = \frac{\frac{\partial C^*(Q,W,R)}{\partial Q}}{\frac{C^*(Q,W,R)}{Q}} = \frac{\text{Marginal Cost}}{\text{Average Cost}} = \frac{1}{3}$$

it follows that marginal cost is less than average cost so there are increasing returns to scale. In fact the production function is homogeneous of degree 3.

Similarly:

$$\frac{\partial \ln \left(C^*\left(Q,W,R\right)\right)}{\partial \ln \left(W\right)} = \frac{\frac{\partial C^*(Q,W,R)}{\partial W}}{\frac{C^*(Q,W,R)}{W}} = \frac{WL^*}{C^*} = \text{ labour's cost share} = \frac{1}{4}$$

it follows that labour's cost share is $\frac{1}{4}$ or 25%.

Problem 95 *Calculate L^*, K^* and λ^* when:*

$$C^*\left(Q,W,R\right) = Q^2\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^3.$$

Answer

We have:

$$\begin{aligned}
L^* &= \frac{\partial C^*(Q,W,R)}{\partial W} = \frac{\partial}{\partial W} Q^2\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^3 \\
&= Q^2\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^2 W^{-\frac{2}{3}} \\
K^* &= \frac{\partial C^*(Q,W,R)}{\partial R} = \frac{\partial}{\partial R} Q^2\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^3 \\
&= Q^2\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^2 R^{-\frac{2}{3}} \\
\lambda^* &= \frac{\partial C^*(Q,W,R)}{\partial Q} = \frac{\partial}{\partial Q} Q^2\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^3 \\
&= 2Q\left(W^{\frac{1}{3}} + R^{\frac{1}{3}}\right)^3.
\end{aligned}$$

The following 3 problems are based on the following information:
Suppose that a cost function is given by:

$$C^*(Q,W,R) = Q\left[W^{\frac{1}{2}} + R^{\frac{1}{2}}\right]^2$$

Problem 96 *Show that $C^*(Q,W,R)$ is homogeneous of degree 1 with respect to W and R.*

Answer

We have:

$$\begin{aligned}
C^*(Q,\lambda W,\lambda R) &= Q\left((\lambda W)^{\frac{1}{2}} + (\lambda R)^{\frac{1}{2}}\right)^2 \\
&= Q\left(\lambda^{\frac{1}{2}}\left((W)^{\frac{1}{2}} + (R)^{\frac{1}{2}}\right)\right)^2 \\
&= \lambda\left(QW^{\frac{1}{2}} + R^{\frac{1}{2}}\right)^2 \\
&= \lambda^1 C^*(Q,W,R).
\end{aligned}$$

Problem 97 *Find $L^*(Q,W,R)$ and $K^*(Q,W,R)$ using Shephard's lemma and show that they are homogeneous of degree 0 with respect to W and R.*

Answer

We have:

$$\begin{aligned} L^*(Q,W,R) &= \frac{\partial C^*(Q,W,R)}{\partial W} = \frac{\partial}{\partial W} Q\left(W^{\frac{1}{2}} + R^{\frac{1}{2}}\right)^2 \\ &= Q\frac{\left(W^{\frac{1}{2}} + R^{\frac{1}{2}}\right)}{W^{\frac{1}{2}}} = Q\left(1 + R^{\frac{1}{2}} W^{-\frac{1}{2}}\right) \\ K^*(Q,W,R) &= \frac{\partial C^*(Q,W,R)}{\partial R} = \frac{\partial}{\partial R} Q\left(W^{\frac{1}{2}} + R^{\frac{1}{2}}\right)^2 \\ &= Q\frac{\left((W)^{\frac{1}{2}} + (R)^{\frac{1}{2}}\right)}{(R)^{\frac{1}{2}}} = Q\left(1 + W^{\frac{1}{2}} R^{-\frac{1}{2}}\right). \end{aligned}$$

Problem 98 *Show that*

$$\frac{\partial L^*(Q,W,R)}{\partial W} < 0.$$

Answer

We have:

$$\begin{aligned} \frac{\partial L^*(Q,W,R)}{\partial W} &= \frac{\partial}{\partial W} Q\left(1 + R^{\frac{1}{2}} W^{-\frac{1}{2}}\right) \\ &= -\frac{1}{2} Q R^{\frac{1}{2}} W^{-\frac{3}{2}} < 0. \end{aligned}$$

The following 3 problems are based on the following information:

Suppose that a cost function is given by:

$$C^*(Q,W,R) = (\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}} Q^\beta$$

where W and R are the prices of inputs L and K, $0 < \alpha < 1$, and $\beta > 0$.

Problem 99 *Show that $C^*(Q,W,R)$ is homogeneous of degree 1 with respect to W and R.*

Answer

We have:

$$\begin{aligned} C^*(Q,\lambda W,\lambda R) &= \left(\delta_1 (\lambda W)^\alpha + \delta_2 (\lambda R)^\alpha\right)^{\frac{1}{\alpha}} Q^\beta \\ &= \left(\lambda^\alpha (\delta_1 W^\alpha + \delta_2 R^\alpha)\right)^{\frac{1}{\alpha}} Q^\beta \\ &= \lambda^1 (\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}} Q^\beta \\ &= C^*(Q,W,R). \end{aligned}$$

Problem 100 *Find $L^*(Q,W,R)$ and $K^*(Q,W,R)$ using Shephard's lemma. Find the marginal cost function and determine the Lagrange multiplier $\lambda^*(Q,W,R)$ for the cost minimization problem.*

Answer

We have:

$$\begin{aligned} L^*(Q,W,R) &= \frac{\partial C^*(Q,W,R)}{\partial W} \\ &= (\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}-1} Q^\beta \times \delta_1 W^{\alpha-1} \\ K^*(Q,W,R) &= \frac{\partial C^*(Q,W,R)}{\partial R} \\ &= (\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}-1} Q^\beta \times \delta_2 R^{\alpha-1} \\ \lambda^*(Q,W,R) &= \frac{\partial C^*(Q,W,R)}{\partial Q} \\ &= \frac{1}{\beta}(\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}} Q^{\beta-1}. \end{aligned}$$

Problem 101 *Find the average cost functions. Suppose there are decreasing returns to scale. What can you say about β then? Is there increasing returns to scale?*

Answer

If there are decreasing returns to scale then $\beta > 1$ since:

$$\begin{aligned} AC\,(Q,W,R) &= \frac{(\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}} Q^\beta}{Q} \\ &= (\delta_1 W^\alpha + \delta_2 R^\alpha)^{\frac{1}{\alpha}} Q^{\beta-1} \end{aligned}$$

so that to have average costs increasing in Q we require the exponent on Q be positive or $\beta > 1$. Similarly if there are increasing returns to scale then average costs must fall with Q so that the exponent on Q must be negative or $\beta < 1$.

Problem 102 *Suppose that an econometrician using data for a firm finds that the cost function is given by:*

$$\ln\left(C^*\left(Q,W,R\right)\right) = 3 + 0.87\ln\left(Q\right) + 0.63\ln\left(W\right) + \delta\ln\left(R\right).$$

What does this tell you about the firm? The econometrician has forgotten to give you the coefficient δ. Can you figure it out on your own or would you need to ask him to find it for you?

Answer

There are increasing returns to scale since the coefficient on $\ln(Q)$ is less than 1 or:

$$\frac{\partial \ln(C^*(Q,W,R))}{\partial \ln(Q)} = \frac{MC}{AC} = 0.87 < 1.$$

Labour's share of costs is:

$$\frac{\partial \ln(C^*(Q,W,R))}{\partial \ln(W)} = \frac{WL^*}{C^*} = 0.63$$

while capital's share of costs is:

$$\delta = \frac{\partial \ln(C^*(Q,W,R))}{\partial \ln(R)} = \frac{WR^*}{C^*} = 1 - 0.63 = 0.37.$$

Problem 103 *Suppose we modify the production function so that* $Q = F(L,K,T)$ *where* T *is the temperature, which is* ***exogenous****. Suppose that* $\frac{\partial F(L,K,T)}{\partial T} > 0$*; that is an increase in temperature always increases output. If* $C^*(Q,W,R,T)$ *is the resulting clever cost function, use the envelope theorem to show that increasing temperature reduces costs or that:*

$$\frac{\partial C^*(Q,W,R,T)}{\partial T} < 0.$$

Answer

The Lagrangian is:

$$\mathcal{L}(\lambda,L,K,Q,W,R,T) = WL + RT + \lambda(Q - F(L,K,T)).$$

From the first-order conditions we have:

$$W = \lambda^* \frac{\partial F(L^*,K^*,T)}{\partial L}.$$

Since $W > 0$ and $\frac{\partial F(L^*,K^*,T)}{\partial L}$ it follows that $\lambda^* > 0$. Now:

$$\begin{aligned} \frac{\partial C^*(Q,W,R,T)}{\partial T} &= \frac{\partial \mathcal{L}(\lambda,L,K,Q,W,R,T)}{\partial T}|_{\lambda=\lambda^*,L=L^*,K=K^*} \\ &= -\lambda \frac{\partial F(L,K,T)}{\partial T}|_{\lambda=\lambda^*,L=L^*,K=K^*} \\ &= -\lambda^* \frac{\partial F(L^*,K^*,T)}{\partial T} < 0 \end{aligned}$$

since $\lambda^* > 0$ and $\frac{\partial F(L^*,K^*,T)}{\partial T} > 0$.

The following 3 problems are based on the following information:

Suppose we have a production function with n inputs $X_1, X_2, \ldots X_n$ with nominal prices: $W_1, W_2, \ldots W_n$ and a Cobb-Douglas production function:

$$Q = F(X_1, X_2, \ldots X_n) = \left(X_1^{\alpha_1} \times X_2^{\alpha_2} \times \cdots \times X_n^{\alpha_n}\right)^k$$

where

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

and $k > 0$. Costs are therefore given by:

$$W_1 X_1 + W_2 X_2 + \cdots + W_n X_n.$$

Let X be the $n \times 1$ vector of the X_i $'s$ and let W be the $n \times 1$ vector of the $W_i's$.
Let X be the $n \times 1$ vector of the $X_i's$ and let W be the $n \times 1$ vector of the $W_i's$.

Problem 104 *Show that k determines the degree of homogeneity of the production function.*

Answer

We have:

$$\begin{aligned} F(\lambda X_1, \lambda X_2, \ldots \lambda X_n) &= \left((\lambda X_1)^{\alpha_1} \times (\lambda X_2)^{\alpha_2} \times \cdots \times (\lambda X_n)^{\alpha_n}\right)^k \\ &= \left(\lambda^{\overbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_n}^{=1}} \left(X_1^{\alpha_1} \times X_2^{\alpha_2} \times \cdots \times X_n^{\alpha_n}\right)\right)^k \\ &= \lambda^k \left(X_1^{\alpha_1} \times X_2^{\alpha_2} \times \cdots \times X_n^{\alpha_n}\right)^k. \end{aligned}$$

Problem 105 *Find the first-order conditions for cost minimization and solve for $X_i^*(Q, W)$ and $\lambda^*(Q, W)$. Show that $\lambda^*(Q, W)$ is homogeneous of degree 1 in W while $X_i^*(Q, W)$ is homogeneous of degree 0.*

Answer

By using the $\ln(\,)$ function we can re-write the production constraint:

$$Q = F(X_1, X_2, \ldots X_n) = \left(X_1^{\alpha_1} \times X_2^{\alpha_2} \times \cdots \times X_n^{\alpha_n}\right)^k$$

as:

$$\frac{1}{k} \ln(Q) - \sum_{i=1}^{n} \alpha_i \ln(X_i) = 0.$$

The Lagrangian is therefore:

$$\begin{aligned} \mathcal{L}(\lambda, X, Q, W) &= W_1 X_1 + W_2 X_2 + \cdots + W_n X_n \\ &\quad + \lambda \left(\frac{1}{k} \ln(Q) - \sum_{i=1}^{n} \alpha_i \ln(X_i)\right). \end{aligned}$$

From the Lagrangian the first-order conditions are:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \Longrightarrow \frac{1}{k}\ln(Q) - \sum_{i=1}^{n}\alpha_i \ln(X_i^*) = 0 \\ \frac{\partial \mathcal{L}}{\partial X_i} &= 0 \Longrightarrow W_i - \frac{\lambda^*\alpha_i}{X_i^*} = 0 \text{ for } i = 1, 2, \ldots n.\end{aligned}$$

We therefore have:

$$X_i^* = \frac{\lambda^*\alpha_i}{W_i}$$

so that substituting this into the first first-order condition from the constraint yields:

$$\begin{aligned}\frac{1}{k}\ln(Q) - \sum_{i=1}^{n}\alpha_i \ln\left(\frac{\lambda^*\alpha_i}{W_i}\right) &= 0 \\ \Longrightarrow \quad & \frac{1}{k}\ln(Q) = \ln(\lambda^*) + \sum_{i=1}^{n}\alpha_i \ln(\alpha_i) - \sum_{i=1}^{n}\alpha_i \ln(W_i)\end{aligned}$$

so that solving for λ^* yields:

$$\lambda^*(Q, W) = \frac{Q^{\frac{1}{k}} W_1^{\alpha_1} \times W_2^{\alpha_2} \times \cdots \times W_n^{\alpha_n}}{\alpha_1^{\alpha_1} \times \alpha_2^{\alpha_2} \times \cdots \times \alpha_n^{\alpha_n}}.$$

Now substituting this into the expression for X_i^* yields:

$$X_i^*(Q, W) = \frac{\lambda^*\alpha_i}{W_i} = \left(\frac{Q^{\frac{1}{k}} W_1^{\alpha_1} \times W_2^{\alpha_2} \times \cdots \times W_n^{\alpha_n}}{\alpha_1^{\alpha_1} \times \alpha_2^{\alpha_2} \times \cdots \times \alpha_n^{\alpha_n}}\right)\frac{\alpha_i}{W_i}.$$

Problem 106 *Show that the firm's cost function is* $C^*(Q, W) = \lambda^*(Q, W)$. *Why isn't* λ^* ***marginal*** *cost here? Verify Shephard's lemma that*

$$\frac{\partial C^*(Q, W)}{\partial W_i} = X_i^*(Q, W).$$

Answer

We have:

$$\begin{aligned}C^*(Q, W) &= W_1 X_1^*(Q, W) + W_2 X_2^*(Q, W) + \cdots + W_n X_n^*(Q, W) \\ &= W_1\frac{\lambda^*\alpha_1}{W_1} + W_2\frac{\lambda^*\alpha_2}{W_2} + \cdots + W_n\frac{\lambda^*\alpha_n}{W_n} \\ &= \lambda^*\left(\underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_n}_{=1}\right) \\ &= \lambda^*(Q, W) = \frac{Q^{\frac{1}{k}} W_1^{\alpha_1} \times W_2^{\alpha_2} \times \cdots \times W_n^{\alpha_n}}{\alpha_1^{\alpha_1} \times \alpha_2^{\alpha_2} \times \cdots \times \alpha_n^{\alpha_n}}.\end{aligned}$$

It is an interesting fact that the Cobb-Douglas production function has a cost function which is of the Cobb-Douglas form.

It is a simple matter then to verify that:

$$\begin{aligned}\frac{\partial C^*(Q,W)}{\partial W_i} &= \frac{\partial \lambda}{\partial W_i}\left(\frac{Q^{\frac{1}{k}} W_1^{\alpha_1}\times W_2^{\alpha_2}\times\cdots\times W_n^{\alpha_n}}{\alpha_1^{\alpha_1}\times\alpha_2^{\alpha_2}\times\cdots\times\alpha_n^{\alpha_n}}\right)\\ &= \frac{Q^{\frac{1}{k}} W_1^{\alpha_1}\times W_2^{\alpha_2}\times\cdots\times W_n^{\alpha_n}}{\alpha_1^{\alpha_1}\times\alpha_2^{\alpha_2}\times\cdots\times\alpha_n^{\alpha_n}}\times\frac{\alpha_i}{W_i}\\ &= X_i^*(Q,W).\end{aligned}$$

The following 3 problems are based on the following information:

Suppose we have a production function with n inputs $X_1, X_2, \ldots X_n$ with nominal prices: $W_1, W_2, \ldots W_n$ and a Constant Elasticity of Substitution (CES) production function:

$$Q = F(X_1, X_2, \ldots X_n) = (\alpha_1 X_1^\rho + \alpha_2 X_2^\rho + \cdots + \alpha_n X_n^\rho)^{\frac{k}{\rho}}$$

where

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

and $k > 0$. Costs are given by:

$$W_1 X_1 + W_2 X_2 + \cdots + W_n X_n.$$

Let X be the $n\times$ 1 vector of the X_i $'s$ and let W be the $n\times$ 1 vector of the W_i $'s$. Let X be the $n\times$ 1 vector of the X_i $'s$ and let W be the $n\times$ 1 vector of the W_i $'s$.

Problem 107 *Show that k determines the degree of homogeneity of the production function and that as $k \to 0$ the CES becomes a Cobb-Douglas production function.*

Answer

Verify that we can rewrite the production relation as:

$$\frac{Q^{\frac{\rho}{k}} - 1}{\rho} = \sum_{i=1}^{n} \alpha_i \frac{X_i^\rho - 1}{\rho}.$$

From L'Hôpital's rule we have:

$$\begin{aligned}\lim_{\rho\to 0}\frac{x^\rho - 1}{\rho} &= \lim_{\rho\to 0}\frac{\frac{d}{d\rho}\left(e^{\rho\ln(x)} - 1\right)}{\frac{d}{d\rho}(\rho)}\\ &= \lim_{\rho\to 0}\frac{\ln(x)\, e^{\rho\ln(x)}}{1}\\ &= \ln(x)\end{aligned}$$

so that as $\rho \to 0$ we have:

$$\begin{aligned} \frac{1}{k}\ln(Q) &= \sum_{i=1}^{n}\alpha_i \ln(X_i) \\ &\Longrightarrow Q = \left(X_1^{\alpha_1} \times X_2^{\alpha_2} \times \cdots \times X_n^{\alpha_n}\right)^k . \end{aligned}$$

Problem 108 *Find the first-order conditions for cost minimization and solve for* $X_i^*(Q,W)$ *and* $\lambda^*(Q,W)$.

Answer

We can re-write the production constraint as:

$$Q^{\frac{\rho}{k}} - \sum_{i=1}^{n}\alpha_i X_i^{\rho} = 0.$$

The Lagrangian is therefore:

$$\begin{aligned} \mathcal{L}(\lambda, X, Q, W) &= W_1X_1 + W_2X_2 + \cdots + W_nX_n \\ &\quad +\lambda\left(Q^{\frac{\rho}{k}} - \sum_{i=1}^{n}\alpha_i X_i^{\rho}\right). \end{aligned}$$

The first-order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \Longrightarrow Q^{\frac{\rho}{k}} - \sum_{i=1}^{n}\alpha_i \left(X_i^*\right)^{\rho} = 0 \\ \frac{\partial \mathcal{L}}{\partial X_i} &= 0 \Longrightarrow W_i - \lambda^*\alpha_i\rho\left(X_i^*\right)^{\rho-1} = 0 \text{ for } i = 1,2,\ldots n. \end{aligned}$$

We therefore have:

$$W_i = \lambda^*\alpha_i\rho\left(X_i^*\right)^{\rho-1}$$

so that taking both sides the power $\frac{\rho}{\rho-1}$ and solving for $\left(X_i^*\right)^{\rho}$ yields:

$$\begin{aligned} \left(X_i^*\right)^{\rho} &= W_i^{\frac{\rho}{\rho-1}}\alpha_i^{-\frac{\rho}{\rho-1}}\rho^{-\frac{\rho}{\rho-1}}\left(\lambda^*\right)^{-\frac{\rho}{\rho-1}} \\ &\Longrightarrow \alpha_i\left(X_i^*\right)^{\rho} = \left(\lambda^*\right)^{-\frac{\rho}{\rho-1}}\alpha_i^{-\frac{1}{\rho-1}}W_i^{\frac{\rho}{\rho-1}}\rho^{-\frac{\rho}{\rho-1}} \\ &\Longrightarrow Q^{\frac{\rho}{k}} = \sum_{i=1}^{n}\alpha_i\left(X_i^*\right)^{\rho} = \left(\lambda^*\right)^{-\frac{\rho}{\rho-1}}\rho^{-\frac{\rho}{\rho-1}}\sum_{i=1}^{n}\alpha_i^{-\frac{1}{\rho-1}}W_i^{\frac{\rho}{\rho-1}} \\ &\Longrightarrow \left(\lambda^*\right)^{-\frac{\rho}{\rho-1}} = \frac{Q^{\frac{\rho}{k}}\rho^{\frac{\rho}{\rho-1}}}{\sum_{i=1}^{n}\alpha_i^{-\frac{1}{\rho-1}}W_i^{\frac{\rho}{\rho-1}}} \end{aligned}$$

and so:

$$\begin{aligned} \left(X_i^*\right)^{\rho} &= \alpha_i^{-\frac{\rho}{\rho-1}}W_i^{\frac{\rho}{\rho-1}}\rho^{-\frac{\rho}{\rho-1}}\left(\lambda^*\right)^{-\frac{\rho}{\rho-1}} \\ &= Q^{\frac{\rho}{k}}\frac{\alpha_i^{-\frac{\rho}{\rho-1}}W_i^{\frac{\rho}{\rho-1}}}{\sum_{i=1}^{n}\alpha_i^{-\frac{1}{\rho-1}}W_i^{\frac{\rho}{\rho-1}}} \end{aligned}$$

so that:

$$\begin{aligned}
X_i^*(Q,W) &= Q^{\frac{1}{k}} \frac{\left(\alpha_i^{-\frac{\rho}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}{\left(\sum_{i=1}^n \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \\
&= Q^{\frac{1}{k}} \frac{\alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{1}{\rho-1}}}{\left(\sum_{i=1}^n \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}.
\end{aligned}$$

Problem 109 *Find the firm's cost function $C^*(Q,W)$. Verify Shephard's lemma that*

$$\frac{\partial C^*(Q,W)}{\partial W_i} = X_i^*(Q,W).$$

Answer

We have:

$$\begin{aligned}
C^*(Q,W) &= \sum_{i=1}^n W_i X_i^*(Q,W) \\
&= \sum_{i=1}^n W_i \frac{Q^{\frac{1}{k}} \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{1}{\rho-1}}}{\left(\sum_{i=1}^n \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \\
&= \frac{Q^{\frac{1}{k}}}{\left(\sum_{i=1}^n \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \left(\sum_{i=1}^n \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^1 \\
&= Q^{\frac{1}{k}} \left(\sum_{i=1}^n \alpha_i^{-\frac{1}{\rho-1}} W_i^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}.
\end{aligned}$$

It is an interesting fact that the CES production function has a cost function which is of the CES form.

The following 3 problems are based on the following information:

Suppose that a firm has a Leontief production function given by:

$$Q = F(L,K) = A\min\left[\sigma^\alpha L^\alpha, K^\alpha\right]$$

where $\min[X,Y]$ is the minimum of X or Y (for example $\min[5,4] = 4$ while $\min[2,7] = 2$) and $\alpha > 0, A > 0$ and $\sigma > 0$.

Problem 110 *Prove that the firm always hires labour and capital so that:*

$$\sigma^\alpha L^\alpha = K^\alpha \;\Leftrightarrow\; \frac{K}{L} = \sigma.$$

Show that the technology is homogeneous of degree α.

Answer

Suppose that $K > \sigma L$. Then $K^\alpha > \sigma^\alpha L^\alpha$ and so the firm would produce:

$$Q = A \min\left[\sigma^\alpha L^\alpha, K^\alpha\right] = A\sigma^\alpha L^\alpha.$$

Now suppose the firm reduced the amount of capital it hires to $K_0 = \sigma L < K$. Then $K_0^\alpha = \sigma^\alpha L^\alpha$ and so:

$$Q = A \min\left[\sigma^\alpha L^\alpha, K_0^\alpha\right] = A\sigma^\alpha L^\alpha$$

and so the firm produces the same amount of output but hires less capital. This cannot be profit maximizing or cost minimizing and so we conclude that: $K \leq \sigma L$.

Now suppose that $K < \sigma L$. Then $K^\alpha < \sigma^\alpha L^\alpha$ and so the firm would produce:

$$Q = A \min\left[\sigma^\alpha L^\alpha, K^\alpha\right] = AK^\alpha.$$

Now suppose the firm reduced the amount of labour it hires to $L_0 = \frac{1}{\sigma}K < L$. Then $\sigma^\alpha L_0^\alpha = K^\alpha$ and so:

$$Q = A \min\left[\sigma^\alpha L_0^\alpha, K^\alpha\right] = AK^\alpha$$

and so the firm produces the same amount of output but hires less labour. This cannot be profit maximizing or cost minimizing and so we conclude that: $K \geq \sigma L$.

Combining the two results: $K \geq \sigma L$ and $K \leq \sigma L$ we conclude that: $K = \sigma L$.

Problem 111 *Show without calculus that the firm's cost function is given by:*

$$C^*(Q, W, R) = (W + \delta R)\, A^{\frac{1}{\alpha}} Q^{\frac{1}{\alpha}}$$

and verify that Shephard's lemma holds for this cost function.

Answer

Since $K = \sigma L$ the firm always hires labour and capital in the same proportion. Because of this we solve for the cost minimizing L^* and K^*, which are independent of W and R, as:

$$\begin{aligned} Q &= A\sigma^\alpha L^\alpha \Longrightarrow L^*(Q) = \frac{1}{\sigma} A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}} \\ Q &= AK^\alpha \Longrightarrow K^*(Q) = A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}} \end{aligned}$$

so that:

$$\begin{aligned} C^*(Q,W,R) &= WL^*(Q) + RK^*(Q) \\ &= W \times \frac{1}{\sigma} A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}} + R \times A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}} \\ &= \left(\frac{W}{\sigma} + R \right) A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}}. \end{aligned}$$

You can then verify that:

$$\begin{aligned} \frac{\partial C^*(Q,W,R)}{\partial W} &= \frac{1}{\sigma} A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}} = L^*(Q) \\ \frac{\partial C^*(Q,W,R)}{\partial R} &= A^{-\frac{1}{\alpha}} Q^{\frac{1}{\alpha}} = K^*(Q) \end{aligned}$$

Chapter 3

Integration and Random Variables

3.1 Summation

Problem 112 *Suppose that* $n = 25$ *and*

$$\begin{aligned} \sum_{i=1}^{n} X_i &= 100, \ \sum_{i=1}^{n} X_i^2 = 500 \\ \sum_{i=1}^{n} Y_i &= 125, \ \sum_{i=1}^{n} Y_i^2 = 850, \ \sum_{i=1}^{n} X_i Y_i = 700. \end{aligned}$$

Based on this information calculate

$$\sum_{i=1}^{n} (3X_i + 4Y_i + 3), \ \sum_{i=1}^{n} (3X_i + 4)^2, \ \sum_{i=1}^{n} (3X_i + 4Y_i)^2$$
$$\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2, \ \sum_{i=1}^{n} \left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right).$$

Now calculate $\hat{\beta}$ *for* $Y_i = \alpha + \beta X_i + e_i$.

Answer

We have:

$$\begin{aligned}
\sum_{i=1}^{n}(3X_i+4Y_i+3) &= 3\sum_{i=1}^{n}X_i+4\sum_{i=1}^{n}Y_i+3n \\
&= 3\times 100+4\times 125+3\times 25=875 \\
\sum_{i=1}^{n}(3X_i+4)^2 &= \sum_{i=1}^{n}\left(9X_i^2+24X_i+16\right) \\
&= 9\sum_{i=1}^{n}X_i^2+24\sum_{i=1}^{n}X_i+16n \\
&= 9\times 500+24\times 100+16\times 25=7300 \\
\sum_{i=1}^{n}(3X_i+4Y_i)^2 &= \sum_{i=1}^{n}\left(9X_i^2+24X_iY_i+16Y_i^2\right) \\
&= 9\sum_{i=1}^{n}X_i^2+24\sum_{i=1}^{n}X_iY_i+16\sum_{i=1}^{n}Y_i^2 \\
&= 9\times 500+24\times 700+16\times 850=34900 \\
\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 &= \sum_{i=1}^{n}\left(X_i^2-2\bar{X}X_i+\bar{X}^2\right) \\
&= \sum_{i=1}^{n}X_i^2-n\bar{X}^2 \\
&= 500-25\left(\frac{100}{25}\right)^2=100 \\
\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right) &= \sum_{i=1}^{n}X_iY_i-n\bar{X}\bar{Y} \\
&= 700-25\left(\frac{100}{25}\right)\left(\frac{125}{25}\right)=200.
\end{aligned}$$

Thus:

$$\hat{\beta}=\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\overline{Y}\right)}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}=\frac{200}{100}=2.$$

Problem 113 *If* $\sum_{i=1}^{7}X_i=35$ *then what is* $\bar{X}$ *and what is* $\sum_{i=1}^{7}(4X_i+2)$? *If* $\sum_{i=1}^{7}X_i^2=255$ *then what is* $\sum_{i=1}^{7}(4X_i+2)^2$ *and what is* $\sum_{i=1}^{7}\left(X_i-\bar{X}\right)^2$*?*

Answer

We have:

$$\begin{aligned}\bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{7}\times 35 = 5.\\ \sum_{i=1}^{7}(4X_i+2) &= 4\sum_{i=1}^{7} X_i + \sum_{i=1}^{7} 2 = 4\times 35 + 7\times 2 = 154.\end{aligned}$$

If $\sum_{i=1}^{7} X_i^2 = 255$ then:

$$\begin{aligned}\sum_{i=1}^{7}(4X_i+2)^2 &= \sum_{i=1}^{7}\left(16X_i^2+16X_i+4\right)\\ &= 16\sum_{i=1}^{7} X_i^2 + 16\sum_{i=1}^{7} X_i + \sum_{i=1}^{7} 4\\ &= 16\times 255 + 16\times 35 + 7\times 4 = 4668.\end{aligned}$$

Also:

$$\begin{aligned}\sum_{i=1}^{7}\left(X_i-\bar{X}\right)^2 &= \sum_{i=1}^{7} X_i - 7\bar{X}^2\\ &= 255 - 7\times 25 = 80.\end{aligned}$$

Problem 114 *If* $\sum_{i=1}^{10} X_i = 50$ *then what is* $\bar{X}$ *and what is* $\sum_{i=1}^{10}(3X_i+2)$? *If* $\sum_{i=1}^{10} X_i^2 = 275$ *then what are* $\sum_{i=1}^{10}(3X_i+2)^2$ *and* $\sum_{i=1}^{10}\left(X_i-\bar{X}\right)^2$?

Answer

We have:

$$\begin{aligned}\bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{10}\times 50 = 5.\\ \sum_{i=1}^{10}(3X_i+2) &= 3\sum_{i=1}^{10} X_i + \sum_{i=1}^{10} 2 = 3\times 50 + 10\times 2 = 170.\end{aligned}$$

If $\sum_{i=1}^{10} X_i^2 = 275$ then:

$$\begin{aligned}\sum_{i=1}^{10}(3X_i+2)^2 &= \sum_{i=1}^{10}\left(9X_i^2+12X_i+4\right)\\ &= 9\sum_{i=1}^{10} X_i^2 + 12\sum_{i=1}^{10} X_i + \sum_{i=1}^{10} 4\\ &= 9\times 275 + 12\times 50 + 10\times 4 = 3115.\end{aligned}$$

Also:

$$\begin{aligned}\sum_{i=1}^{10}\left(X_i-\bar{X}\right)^2 &= \sum_{i=1}^{10}X_i-10\bar{X}^2\\ &= 275-10\times 5^2=25.\end{aligned}$$

3.2 Integration

Problem 115 *What is the anti-derivative of e^{-x}? Show that:*

$$\int_0^\infty e^{-x}dx=1-e^{-T}.$$

Answer

The anti-derivative of e^{-x} is $-e^{-x}$ so that:

$$\int_0^\infty e^{-x}dx=-e^{-x}|_{x=0}^T=1-e^{-T}.$$

Problem 116 *If $G_n=\int_o^T x^n e^{-x}dx$ use integration by parts to derive a relationship between G_n and G_{n-1} for $n>0$. What then is G_2?*

Answer

Using integration by parts with:

$$\begin{array}{cc} u=x^n & v'=e^{-x}\\ u'=nx^{n-1} & v=-e^{-x}\end{array}$$

we have for $n>0$:

$$\begin{aligned}G_n &= \int_0^\infty x^n e^{-x}dx\\ &= -x^n e^{-x}|_{x=0}^T+n\int_0^\infty x^{n-1}e^{-x}dx\\ &= -T^n e^{-T}+nG_{n-1}.\end{aligned}$$

Thus

$$\begin{aligned}G_2 &= -T^2e^{-T}+2G_1\\ &= -T^2e^{-T}+2\left(-Te^{-T}+G_0\right).\end{aligned}$$

since:

$$G_1=-T^1e^{-T}+1G_0=-Te^{-T}+G_0.$$

But by the previous problem

$$G_0 = \int_0^\infty e^{-x} dx = 1 - e^{-T}$$

so that:

$$\begin{aligned} G_2 &= -T^2 e^{-T} + 2\left(-Te^{-T} + \left(1 - e^{-T}\right)\right) \\ &= -e^{-T}\left(T^2 + 2T + 1\right) + 2. \end{aligned}$$

Problem 117 *If* $\Gamma(n, \alpha) = \int_0^\infty x^{n-1} e^{-\alpha x} dx$ *for* $\alpha > 0$ *use integration by parts to derive a relationship between* $\Gamma(n, \alpha)$ *and* $\Gamma(n-1, \alpha)$ *for* $n > 0$*. Show that for* n *a positive integer that:*

$$\Gamma(n, \alpha) = \frac{1}{\alpha^n}(n-1)!.$$

Answer

Using integration by parts with:

$$\begin{array}{cc} u = x^{n-1} & v' = e^{-\alpha x} \\ u' = nx^{n-2} & v = -\frac{1}{\alpha} e^{-\alpha x} \end{array}$$

we have for $n > 0$:

$$\begin{aligned} \Gamma(n, \alpha) &= \int_0^\infty x^{n-1} e^{-\alpha x} dx \\ &= -\frac{1}{\alpha} x^{n-1} e^{-\alpha x}\Big|_{x=0}^\infty + \frac{n-1}{\alpha} \int_0^\infty x^{n-2} e^{-\alpha x} dx \\ &= \frac{n-1}{\alpha} \Gamma(n-1, \alpha). \end{aligned}$$

For $n = 1$ we have $\Gamma(n, \alpha) = \frac{1}{\alpha^n}(n-1)!$ since:

$$\Gamma(1, \alpha) = \int_0^\infty e^{-\alpha x} dx = \frac{1}{\alpha} = \frac{0!}{\alpha^1}$$

and if the statement is true for $n - 1$ so that: $\Gamma(n-1, \alpha) = \frac{(n-2)!}{\alpha^{n-1}}$ then:

$$\begin{aligned} \Gamma(n, \alpha) &= \frac{n-1}{\alpha} \Gamma(n-1, \alpha) \\ &= \frac{n-1}{\alpha} \frac{(n-2)!}{\alpha^{n-1}} = \frac{(n-1)!}{\alpha^n} \end{aligned}$$

and so the result follows by induction.

Problem 118 *Calculate the following integrals:*

$$i) \int_1^5 x^2 dx,$$
$$ii) \int_0^3 e^{-x} dx,$$
$$iii) \int_0^3 \left(e^{-x}+2\right)^2 dx,$$
$$iv) \int_0^3 xe^{-x} dx,$$
$$v) \int_0^3 x^2 e^{-x} dx.$$

Answer

We have:

$$\begin{aligned}
\int_1^5 x^2 dx &= \frac{x^3}{3}|_{x=1}^5 = \frac{5^5}{3} - \frac{1^5}{3} = \frac{124}{3}, \\
\int_0^3 e^{-x} dx &= 1 - e^{-3} \\
\int_0^3 \left(e^{-x}+2\right)^2 dx &= \int_0^3 \left(e^{-2x} + 4e^{-x} + 4\right) dx \\
&= \int_0^3 e^{-2x} + 4\int_0^3 e^{-x} + 4\int_0^3 dx \\
&= -\frac{1}{2}e^{-6} + \frac{33}{2} - 4e^{-3} \\
\int_0^3 xe^{-x} dx &= -4e^{-3} + 1 \\
\int_0^3 x^2 e^{-x} dx &= -17e^{-3} + 2.
\end{aligned}$$

Problem 119 *Calculate*

$$\int_0^T te^{-rt} dt$$

using integration by parts. From this find:

$$\int_0^\infty te^{-rt} dt$$

assuming that $r > 0$.

Answer

Using integration by parts with:

$$\begin{array}{ll} u = t & v' = e^{-rt} \\ u' = 1 & v = -\frac{e^{-rt}}{r} \end{array}$$

we have:

$$\begin{aligned}
\int_0^T te^{-rt}dt &= -\frac{e^{-rt}}{r} \times t|_{t=0}^T + \frac{1}{r}\int_0^T e^{-rt}dt \\
&= -\frac{e^{-Tr}}{r}T + \frac{1}{r}\int_0^T e^{-rt}dt \\
&= -\frac{e^{-Tr}}{r}T - \frac{1}{r}\frac{e^{-rt}}{r}|_{t=0}^T \\
&= -\frac{e^{-Tr}}{r}T + \frac{1}{r^2}\left(1 - e^{-rT}\right).
\end{aligned}$$

Since $r > 0$ it follows that $e^{-rT} \to 0$ and $e^{-rT}T \to 0$ so that:

$$\int_0^\infty te^{-rt}dt = \frac{1}{r^2}.$$

Problem 120 *Using integration by parts, calculate:*

$$\int_0^T t^2e^{-rt}dt, \text{ where } r > 0$$

and find the limit of this expression as $T \to \infty$ when $r > 0$.

Answer

Letting $u = t^2$ and $v' = e^{-rt}$ we have:

$$\begin{array}{ll} u = t^2 & v' = e^{-rt} \\ u' = 2t & v = -\frac{e^{-rt}}{r} \end{array}$$

so that using integration by parts we have:

$$\begin{aligned}
\int_0^T t^2e^{-rt}dt &= -\frac{e^{-rt}}{r}t^2|_{t=0}^T + \frac{2}{r}\int_0^T te^{-rt}dt \\
&= -\frac{e^{-Tr}}{r}T^2 + \frac{2}{r}\int_0^T te^{-rt}dt.
\end{aligned}$$

The new integral was found in the last problem so that:

$$\begin{aligned}\int_0^T t^2 e^{-rt} dt &= -\frac{e^{-Tr}}{r} T^2 + \frac{2}{r} \int_0^T t e^{-rt} dt \\ &= -\frac{e^{-Tr}}{r} T^2 + \frac{2}{r} \left(\frac{1}{r^2} \left(1 - e^{-rT}\right) - \frac{T}{r} e^{-Tr}. \right).\end{aligned}$$

Letting $T \to \infty$ we have:

$$\int_0^\infty t^2 e^{-rt} dt = \frac{2}{r^3}.$$

Problem 121 *Using integration by parts find:*

$$\int_0^5 x e^{-2x} dx.$$

The incomplete gamma function is given by: $\Gamma(n, m) = \int_m^\infty x^{n-1} e^{-x} dx$ *for* $m \geq 0$. *Use integration by parts to find the relationship between* $\Gamma(n, m)$ *and* $\Gamma(n-1, m)$.

Answer

Using integration by parts with:

$$\begin{array}{ll} u = x & v' = e^{-2x} \\ u' = 1 & v = -\frac{e^{-2x}}{2} \end{array}$$

we have:

$$\begin{aligned}\int_0^5 x e^{-2x} dx &= -\frac{x e^{-2x}}{2} |_{x=0}^5 + \frac{1}{2} \int_0^T e^{-2x} dt \\ &= -\frac{5}{2} e^{-10} + \frac{1}{2} \left(-\frac{e^{-2x}}{2} |_{x=0}^5 \right) \\ &= -\frac{5}{2} e^{-10} + \frac{1}{4} \left(1 - e^{-10}\right) \\ &= \frac{1}{4} - \frac{11}{4} e^{-10}.\end{aligned}$$

Now given:

$$\begin{array}{rl} u = x^{n-1} & v' = e^{-x} \\ u' = (n-1) x^{n-2} & v = -e^{-x} \end{array}$$

we have:

$$\begin{aligned}\Gamma(n,m) &= \int_m^\infty x^{n-1}e^{-x}dx \\ &= -x^{n-1}e^{-x}|_{x=m}^\infty + (n-1)\int_m^\infty x^{n-2}e^{-x}dx \\ &= m^{n-1}e^{-m} + (n-1)\Gamma(n-1,m).\end{aligned}$$

Problem 122 *If $L^*(Q,W,R)$ is the conditional factor demand for labour, what is: $\int_{W_1}^{W_2} L^*(Q,W,R)\,dW$ equal to?*

Answer

By Shephard's lemma we have:

$$L^*(Q,W,R) = -\frac{\partial C^*(Q,W,R)}{\partial W}$$

so that the anti-derivative of $L^*(Q,W,R)$ (with respect to W) is $-C^*(Q,W,R)$. Thus:

$$\begin{aligned}\int_{W_1}^{W_2} L^*(Q,W,R)\,dW &= \int_{W_1}^{W_2} -\frac{\partial C^*(Q,W,R)}{\partial W}dW \\ &= -C^*(Q,W,R)|_{W=W_1}^{W_2} \\ &= C^*(Q,W_1,R) - C^*(Q,W_2,R).\end{aligned}$$

Problem 123 *Consider the standard normal density:*

$$p(x) = e^{-x^2/2}/\sqrt{2\pi}, \quad -\infty < x < \infty$$

It is a fact that:$\int_{-\infty}^{\infty} p(x)\,dx = 1$ and that if the random variable X has this density then $E[X] = 0$. Show that $Var[X] = 1$ (hint: use integration by parts with $u = x, v' = xp(x)$ and show that $v = -p(x)$. Now show that $E\left[X^4\right] = 3$ again using integration by parts (hint use $u = x^3$ and the same v' as before). The value 3 is the kurtosis of the standard normal distribution, a fact commonly used in applied work to test if data are consistent with the hypothesis of a normal distribution.

Answer

Using integration by parts with:

$$\begin{aligned} u &= x & v' &= x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \\ u' &= 1 & v &= -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\end{aligned}$$

we have:

$$\begin{aligned} Var[X] &= \int_{-\infty}^{\infty} (x - E[X])^2 p(x)\,dx \\ &= \int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= -x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Big|_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= 1 \end{aligned}$$

since:

$$\lim_{x\to\pm\infty} xe^{-\frac{x^2}{2}} = 0$$

using L'Hôpital's rule since:

$$\lim_{x\to\pm\infty} xe^{-\frac{x^2}{2}} = \lim_{x\to\pm\infty} \frac{x}{e^{\frac{x^2}{2}}} = \lim_{x\to\pm\infty} \frac{1}{xe^{\frac{x^2}{2}}} = 0$$

and

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} p(x)\,dx = 1.$$

Now using integration by parts with:

$$\begin{array}{ll} u = x^3 & v' = x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \\ u' = 3x^2 & v = -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \end{array}$$

we have:

$$\begin{aligned} E\left[X^4\right] &= \int_{-\infty}^{\infty} x^4 p(x)\,dx \\ &= \int_{-\infty}^{\infty} x^4 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= -x^3\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Big|_{x=-\infty}^{\infty} + 3\int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= 3Var[X] = 3 \end{aligned}$$

since $Var[X] = 1$ and:

$$\lim_{x\to\pm\infty} x^3 e^{-\frac{x^2}{2}} = 0$$

(using L'Hôpital's rule since:

$$\begin{aligned}\lim_{x\to\pm\infty} x^3 e^{-\frac{x^2}{2}} &= \lim_{x\to\pm\infty} \frac{x^3}{e^{\frac{x^2}{2}}} = \lim_{x\to\pm\infty} \frac{3x^2}{xe^{\frac{x^2}{2}}} \\ &= \lim_{x\to\pm\infty} \frac{6x}{\left(x^2+1\right) e^{\frac{x^2}{2}}} = \lim_{x\to\pm\infty} \frac{6}{\left(x^3+3x\right) e^{\frac{x^2}{2}}} = 0).\end{aligned}$$

Problem 124 *The n^{th} uncentred moment is defined as: $\mu_n \equiv E\left[X^n\right]$. For the standard normal distribution use proof by induction to show that: $\mu_n = 0$ for n odd and that for n even*

$$\mu_n = (n-1) \times (n-3) \times \cdots \times 3 \times 1$$

Calculate μ_n for $n = 0, 1, 2, 3, 4, 5, 6$.

Answer

Using integration by parts with:

$$\begin{aligned} u &= x^{n-1} & v' &= x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \\ u' &= (n-1)\, x^{n-2} & v &= -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\end{aligned}$$

we have for $n > 0$:

$$\begin{aligned}\mu_n &= \int_{-\infty}^{\infty} x^n \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= -x^{n-1}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Big|_{x=-\infty}^{\infty} + (n-1)\int_{-\infty}^{\infty} x^{n-2}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= (n-1)\,\mu_{n-2}\end{aligned}$$

since:

$$\lim_{x\to\pm\infty} x^{n-1} e^{-\frac{x^2}{2}} = 0.$$

We now use a proof by induction to show that $\mu_n = 0$ for n odd. We know that $\mu_1 = E[X] = 0$. Now assume that $\mu_k = 0$ for $k < n$ and k odd. In particular if n is odd then so too is $n-2$ and so μ_{n-2} is also odd so that by the induction hypothesis $\mu_{n-2} = 0$ and so:

$$\mu_n = (n-1)\,\mu_{n-2} = 0$$

so that: $\mu_n = 0$ for all n odd.

For n even we have:

$$\mu_0 = E\left[X^0\right] = 1.$$

Since if n is even so too is $n-2$ we have from the induction hypothesis that:

$$\mu_{n-2} = (n-3) \times \cdots \times 3 \times 1$$

so that:

$$\begin{aligned} \mu_n &= (n-1) \times \mu_{n-2} \\ &= (n-1) \times (n-3) \times \cdots \times 3 \times 1. \end{aligned}$$

We thus have:

$n =$	0	1	2	3	4	5	6
$\mu_n =$	1	0	1	0	3	0	15

3.3 Random Variables

The following 3 problems are based on the following information:

Consider an investor who can invest in three assets with returns R_0, R_1 and R_2 where:

$$\begin{aligned} E[R_0] &= 3, Var[R_1] = 0 \\ E[R_1] &= 10, Var[R_1] = 25 \\ E[R_2] &= 7, Var[R_2] = 9, Cov[R_1, R_2] = -9. \end{aligned}$$

Problem 125 *What kind of asset does R_0 correspond to? Calculate $E\left[R_1^2\right]$, $E[R_1 R_2]$ and the correlation coefficient: ρ between R_1 and R_2.*

Answer

R_0 has a variance of 0 and hence is a degenerate random variable; that is $R_0 = 3$ with probability 1. It is therefore a riskless asset (something like Canada savings bonds).

We have:

$$\begin{aligned} E\left[R_1^2\right] &= E[R_1]^2 + Var[R_1] = 10^2 + 25 = 125 \\ E[R_1 R_2] &= E[R_1] E[R_2] + Cov[R_1, R_2] = 7 \times 10 + -9 = 61 \\ \rho &= \frac{Cov[R_1, R_2]}{\sqrt{Var[R_1]}\sqrt{Var[R_1]}} = \frac{-9}{\sqrt{9}\sqrt{16}} = -\frac{3}{4} \end{aligned}$$

Problem 126 *The return on the portfolio is: $R = R_0\omega_o + \omega_1 R_1 + \omega_2 R_2$ where: $1 = \omega_o + \omega_1 + \omega_2$. If $\omega_o = 0.3, \omega_1 = 0.5$ and $\omega_2 = 0.2$ calculate $E[R]$, $Var[R]$, and $E\left[R^2\right]$. If $\omega_0 = 0$ find the portfolio with the least risk.*

Answer

We have:

$$
\begin{aligned}
E[R] &= 0.3E[R_0] + 0.5E[R_1] + 0.2E[R_2] \\
&= 0.3 \times 3 + 0.5 \times 10 + 0.2 \times 7 = 7.3
\end{aligned}
$$

and using the fact that $R_0 = 3$ is a constant and therefore does not affect the variance we have:

$$
\begin{aligned}
Var[R] &= Var[0.3 \times R_0 + 0.3R_1 + 0.5R_2] \\
&= Var[0.3R_1 + 0.5R_2] \\
&= (0.3)^2 Var[R_1] + (0.5)^2 Var[R_2] + 2 \times 0.5 \times 0.3 \times Cov[R_1, R_2] \\
&= (0.3)^2 \times 25 + (0.5)^2 \times 9 + 2 \times 0.5 \times 0.3 \times -9 \\
&= 1.8.
\end{aligned}
$$

If $\omega_0 = 0$ then since the asset weights sum to 1 we have:

$$1 = \omega_o + \omega_1 + \omega_2 \Rightarrow \omega_2 = 1 - \omega_1$$

and so replacing ω_2 with $1 - \omega_1$ in

$$
\begin{aligned}
Var[R] &= Var[\omega_o R_0 + \omega_1 R_1 + \omega_2 R_2] \\
&= Var[\omega_1 R_1 + \omega_2 R_2] \\
&= Var[\omega_1 R_1 + (1 - \omega_1) R_2] \\
&= \omega_1^2 Var[R_1] + (1 - \omega_1)^2 Var[R_2] + 2\omega_1 (1 - \omega_1) Cov[R_1, R_2] \\
&= (\omega_1)^2 25 + (1 - \omega_1)^2 9 + 2 \times \omega_1 \times (1 - \omega_1) \times (-9)
\end{aligned}
$$

we have:

$$
\begin{aligned}
f(\omega_1) &= Var[R] = (\omega_1)^2 25 + (1 - \omega_1)^2 9 + 2 \times \omega_1 \times (1 - \omega_1) \times (-9) \\
&\Rightarrow f'(\omega_1) = 50\omega_1 - 18(1 - \omega_1) - 18(1 - 2\omega_1) \\
&\Longrightarrow f'(\omega_1^*) = 0 = 50\omega_1^* - 18(1 - \omega_1^*) - 18(1 - 2\omega_1^*) \\
&\Longrightarrow \omega_1^* = \frac{9}{26},\ \omega_2^* = 1 - \frac{9}{26} = \frac{17}{26}.
\end{aligned}
$$

This is a global minimum since:

$$f''(\omega_1) = 50 + 18 + 36 = 104 > 0$$

so that $f(\omega_1)$ is globally convex.

Problem 127 *If the investor wants an expected return of return of* 8, *what portfolio does this with the minimum amount of risk? What is the variance of this portfolio.*

Answer

Since the portfolio needs to have an expected return of 8 and the weights of the portfolio sum to 1 we have:

$$\begin{aligned} E[R] &= 8 = \omega_o 3 + \omega_1 10 + \omega_2 7 \\ 1 &= \omega_o + \omega_1 + \omega_2 \end{aligned}$$

or

$$8 = (1 - \omega_1 - \omega_2)\, 3 + \omega_1 10 + \omega_2 7$$

or:

$$5 = \omega_1 7 + \omega_2 4$$

which will act as the constraint. The variance of the portfolio is:

$$Var[R] = 25\omega_1^2 + 9\omega_2^2 - 18\omega_1\omega_2.$$

We will minimize $\frac{Var[R]}{2}$ subject to the above constraint so that the Lagrangian is:

$$L(\lambda, \omega_1, \omega_2) = \frac{25\omega_1^2 + 9\omega_2^2 - 18\omega_1\omega_2}{2} + \lambda(5 - \omega_1 7 - \omega_2 4)$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial L(\lambda^*, \omega_1^*, \omega_2^*)}{\partial \lambda} &= 5 - 7\omega_1^* - 4\omega_2^* = 0 \\ \frac{\partial L(\lambda^*, \omega_1^*, \omega_2^*)}{\partial \omega_1} &= 25\omega_1^* - 9\omega_2^* - 7\lambda^* = 0 \\ \frac{\partial L(\lambda^*, \omega_1^*, \omega_2^*)}{\partial \omega_2} &= -9\omega_1^* + 9\omega_2^* - 4\lambda^* = 0. \end{aligned}$$

with second-order conditions being satisfied since:

$$H = \det \begin{bmatrix} 0 & -7 & -4 \\ -7 & 25 & -9 \\ -4 & -9 & 9 \end{bmatrix} = -1345 < 0.$$

The first-order conditions in matrix notation are then:

$$\begin{bmatrix} 0 & -7 & -4 \\ -7 & 25 & -9 \\ -4 & -9 & 9 \end{bmatrix} \begin{bmatrix} \lambda^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}$$

so that solving (by Cramer's rule or matrix inversion) we have:

$$\begin{bmatrix} \lambda^* \\ \omega_1^* \\ \omega_2^* \end{bmatrix} = \begin{bmatrix} 0 & -7 & -4 \\ -7 & 25 & -9 \\ -4 & -9 & 9 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{144}{269} \\ \frac{99}{269} \\ \frac{163}{269} \end{bmatrix}.$$

Thus the optimal portfolio has:

$$\omega_1^* = \frac{99}{269} = 0.37, \ \omega_2^* = \frac{163}{269} = 0.61$$

or 37% in R_1, and 61% in R_2 and

$$\omega_o^* = 1 - \omega_1^* - \omega_2^* = \frac{7}{269} = 0.02$$

or 2% in R_0. This is a global minimum since the objective function is convex (prove this!) and the constraint is linear.

The variance of the portfolio is then given by:

$$\begin{aligned} Var[R] &= 25\omega_1^2 + 9\omega_2^2 - 18\omega_1\omega_2 \\ &= 25\left(\frac{99}{269}\right)^2 + 9\left(\frac{163}{269}\right)^2 - 18 \times \frac{99}{269} \times \frac{163}{269} \\ &= 2.6766. \end{aligned}$$

Problem 128 *Let R_1, R_2 and R_3 be three random variables having the following properties:*

$$\begin{aligned} E[R_1] &= 7, E[R_2] = 10, E[R_3] = 3 \\ Var[R_1] &= 9, Var[R_2] = 16, Cov[R_1, R_2] = -6, Var[R_3] = 0. \end{aligned}$$

What is $E\left[R_1^2\right]$, $E[R_1 R_2]$ and the correlation coefficient between R_1 and R_2? What kind of random variable is R_3? If

$$R = 0.3R_1 + 0.5R_2 + 0.2R_3,$$

what is $E[R]$ and $Var[R]$?

Answer

We have:

$$\begin{aligned} E\left[R_1^2\right] &= E[R_1]^2 + Var[R_1] = 7^2 + 9 = 58 \\ E[R_1 R_2] &= E[R_1]E[R_2] + Cov[R_1, R_2] = 7 \times 10 + -6 = 64 \\ \rho &= \frac{Cov[R_1, R_2]}{\sqrt{Var[R_1]}\sqrt{Var[R_1]}} = \frac{-6}{\sqrt{9}\sqrt{16}} = -\frac{1}{2} \end{aligned}$$

Since $Var[R_3] = 0$ it follows that R_3 is a degenerate random variable or $R_3 = 3$.

For R

$$\begin{aligned} E[R] &= 0.3E[R_1] + 0.5E[R_2] + 0.2E[R_3] \\ &= 0.3 \times 7 + 0.5 \times 10 + 0.2 \times 3 = 7.7. \end{aligned}$$

Since $R_3 = 3$ is a constant we have:

$$\begin{aligned} Var[R] &= Var[0.3R_1 + 0.5R_2 + 0.2 \times 3] \\ &= Var[0.3R_1 + 0.5R_2] \\ &= (0.3)^2 Var[R_1] + (0.5)^2 Var[R_2] + 2 \times 0.5 \times 0.3 Cov[R_1, R_2] \\ &= (0.3)^2 \times 9 + (0.5)^2 \times 16 + 2 \times 0.5 \times 0.3 \times -6 \\ &= 3.01. \end{aligned}$$

Problem 129 *For any two random variables: X and Y, what value of a would minimize $Var[Y - aX]$?*

Answer

We have:

$$\begin{aligned} f(a) &= Var[Y - aX] \\ &= Var[Y] + a^2 Var[X] - 2a Cov[X, Y] \end{aligned}$$

so that:

$$\begin{aligned} f'(a^*) &= 0 \Longrightarrow 2a^* Var[X] - 2Cov[X, Y] = 0 \\ &\Longrightarrow a^* = \frac{Cov[X, Y]}{Var[X]}. \end{aligned}$$

This is a global minimum since:

$$f''(a) = 2Var[X] > 0$$

so that $f(a)$ is globally convex.

Problem 130 *Consider a barrel with $30,000$ dice in it, let X_i be the outcome of the i^{th} die and let S be the sum of all the dice. Each of the dice is fair so that the probabilities of $1, 2, 3, 4, 5$ and 6 are all $\frac{1}{6}$. Calculate $E[X_i]$, $E[X_i^2]$, $Var[X_i]$, $E[S]$ and $Var[S]$. It is a fact that 95% of the die rolls will fall in a range*

$$E[S] \pm 1.96\sqrt{Var[S]}.$$

Calculate this range and compare this with the range of all possible values.

Answer

We have:

$$\begin{aligned} E[X_i] &= \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = \frac{7}{2}. \\ E[X_i^2] &= \frac{1}{6} \times 1^2 + \frac{1}{6} \times 2^2 + \frac{1}{6} \times 3^2 + \frac{1}{6} \times 4^2 + \frac{1}{6} \times 5^2 + \frac{1}{6} \times 6^2 = \frac{91}{6} \\ Var[X_i] &= E[X_i^2] - E[X_i]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}. \end{aligned}$$

If $S = X_1 + X_2 + \cdots + X_{20000}$ then:

$$\begin{aligned} E[S] &= E[X_1 + X_2 + \cdots + X_{30000}] \\ &= E[X_1] + E[X_2] + \cdots + E[X_{30000}] \\ &= \frac{7}{2} + \frac{7}{2} + \cdots + \frac{7}{2} = 30000 \times \frac{7}{2} = 105000 \end{aligned}$$

and

$$\begin{aligned} Var[S] &= Var[X_1 + X_2 + \cdots + X_{30000}] \\ &= 1^2 Var[X_1] + 1^2 Var[X_2] + \cdots + 1^2 Var[X_{30000}] \\ &= \frac{35}{12} + \frac{35}{12} + \cdots + \frac{35}{12} = 30000 \times \frac{35}{12} = 87500. \end{aligned}$$

We therefore have 95% of the outcomes of S falling in the range:

$$E[S] \pm 1.96\sqrt{Var[S]} = 105000 \pm 1.96\sqrt{87500} = 105000 \pm 580.$$

The lowest possible value of S occurs when each die is a 1 which results in $S = 30000$, while the highest possible value of S occurs when each die is a 6 which results in $S = 180000$. Thus the total range of S is $180000 - 30000 = 150\,000$. Despite this very large range, almost all values of S will in a very narrow band around 90000; for example 95% of the outcomes will occur in the range 105000 ± 580. This is an illustration of the law of large numbers.

Problem 131 *Consider a barrel with* $20,000$ *dice in it, let* X_i *be the outcome of the* i^{th} *die and let* S *be the sum of all the dice. Each of the dice is crooked so that the probabilities of* $1, 2, 3, 4, 5$ *and* 6 *are respectively:* $\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}$ *and* $\frac{1}{2}$. *Calculate* $E[X_i]$, $E[X_i^2]$, $Var[X_i]$, $E[S]$ *and* $Var[S]$. *It is a fact that* 95% *of the die rolls will fall in a range*

$$E[S] \pm 1.96\sqrt{Var[S]}.$$

Calculate this range and compare this with the range of all possible values.

Answer

We have:

$$\begin{aligned} E[X_i] &= \frac{1}{10} \times 1 + \frac{1}{10} \times 2 + \frac{1}{10} \times 3 + \frac{1}{10} \times 4 + \frac{1}{10} \times 5 + \frac{1}{2} \times 6 = \frac{9}{2}. \\ E[X_i^2] &= \frac{1}{10} \times 1^2 + \frac{1}{10} \times 2^2 + \frac{1}{10} \times 3^2 + \frac{1}{10} \times 4^2 + \frac{1}{10} \times 5^2 + \frac{1}{2} \times 6^2 = \frac{47}{2} \\ Var[X_i] &= E[X_i^2] - E[X_i]^2 = \frac{47}{2} - \left(\frac{9}{2}\right)^2 = \frac{13}{4}. \end{aligned}$$

If $S = X_1 + X_2 + \cdots + X_{20000}$ then:

$$\begin{aligned} E[S] &= E[X_1 + X_2 + \cdots + X_{20000}] \\ &= E[X_1] + E[X_2] + \cdots + E[X_{20000}] \\ &= \frac{9}{2} + \frac{9}{2} + \cdots + \frac{9}{2} = 20000 \times \frac{9}{2} = 90000 \end{aligned}$$

and

$$\begin{aligned} Var[S] &= Var[X_1 + X_2 + \cdots + X_{20000}] \\ &= 1^2 Var[X_1] + 1^2 Var[X_2] + \cdots + 1^2 Var[X_{20000}] \\ &= \frac{13}{4} + \frac{13}{4} + \cdots + \frac{13}{4} = 20000 \times \frac{13}{4} = 65000. \end{aligned}$$

We therefore have 95% of the outcomes of S falling in the range:

$$E[S] \pm 1.96\sqrt{Var[S]} = 90000 \pm 1.96\sqrt{65000} = 90000 \pm 500.$$

The lowest possible value of S occurs when each die is a 1 which results in $S = 20000$, while the highest possible value of S occurs when each die is a 6 which results in $S = 120000$. Thus the total range of S is $120000 - 20000 = 100000$. Despite this very large range, almost all values of S will in a very narrow band around 90000; for example 95% of the outcomes will occur in the range 90000 ± 500. This is an illustration of the law of large numbers.

Problem 132 *If $Z \sim N[0,1]$ it is a fact that $E[Z^2] = 1$ and $E[Z^4] = 3$. Note that if $Y = Z^2$ then $Y \sim \chi_1^2$ and so we have: $E[Y] = 1$ and $E[Y^2] = 3$ or*

$$Var[Y] = E[Y^2] - E[Y]^2 = 3 - 1 = 2.$$

Consider generalizing this to the case where if $Y \sim \chi_r^2$ then $E[Y] = r$ and $Var[Y] = 2r$. (see the lecture notes for the relationship between the standard normal and chi-squared distributions).

Answer

If

$$Y = Z_1^2 + Z_2^2 + \cdots + Z_r^2$$

where: $Z_i \sim N[0,1]$ are independent then $Y \sim \chi_r^2$. We therefore have:

$$\begin{aligned} E[Y] &= E[Z_1^2 + Z_2^2 + \cdots + Z_r^2] \\ &= E[Z_1^2] + E[Z_2^2] + \cdots + E[Z_r^2] \\ &= 1 + 1 + \cdots + 1 = r. \end{aligned}$$

Now

$$\begin{aligned} Var[Z_i^2] &= E\left[\left(Z_i^2\right)^2\right] - E[Z_i^2]^2 \\ &= E[Z_i^4] - E[Z_i^2]^2 \\ &= 3 - 1 = 2 \end{aligned}$$

so that since the $Z_i's$ are independent:

$$\begin{aligned} Var[Y] &= Var[Z_1^2 + Z_2^2 + \cdots + Z_r^2] \\ &= 1^2 Var[Z_1^2] + 1^2 Var[Z_2^2] + \cdots + 1^2 Var[Z_r^2] \\ &= 2 + 2 + \cdots + 2 = 2r. \end{aligned}$$

Problem 133 *Let X and W be two random variables having the following properties:*

$$\begin{aligned} E[X] &= 5,\ Var[X] = 9, \\ E[W] &= 10,\ Var[W] = 16, \\ Cov[X,W] &= -4. \end{aligned}$$

What is $E\left[X^2\right]$, $E[XW]$, $Cov[W,W]$, $Cov[W,5]$ and the correlation coefficient: ρ between X and W? If

$$Y = 20 + 2X - 5W,$$

what is $E[Y]$ and $Var[Y]$? What is $Cov[Y,X]$?

Answer

We have:

$$\begin{aligned} E\left[X^2\right] &= E[X]^2 + Var[X] = 5^2 + 9 = 34 \\ E[XW] &= E[X]E[W] + Cov[X,W] = 5 \times 10 + -4 = 46 \\ Cov[W,W] &= Var[W] = 16 \\ Cov[W,5] &= 0 \\ \rho &= \frac{Cov[X,W]}{\sqrt{Var[X]}\sqrt{Var[W]}} = \frac{-4}{\sqrt{9}\sqrt{16}} = -\frac{1}{3} \end{aligned}$$

For Y

$$\begin{aligned} E[Y] &= 20 + 2E[X] - 5E[W] \\ &= 20 + 2 \times 5 - 5 \times 10 = -20 \\ Var[Y] &= 2^2 Var[X] + (-5)^2 Var[W] + 2 \times (2) \times (-5) Cov[X,W] \\ &= 2^2 \times 9 + (-5)^2 \times 16 + 2 \times (2) \times (-5) \times (-4) = 516. \end{aligned}$$

For $Cov[Y,X]$ we have:

$$\begin{aligned} Cov[Y,X] &= E[(Y - E[Y]) \times (X - E[X])] \\ &= E[(20 + 2X - 5W - E[20 + 2X - 5W]) \times (X - E[X])] \\ &= E[2(X - E[X]) - 5(W - E[W]) \times (X - E[X])] \\ &= 2E[(X - E[X]) \times (X - E[X])] + 5E[(W - E[W]) \times (X - E[X])] \\ &= 2E\left[(X - E[X])^2\right] - 5E[(W - E[W]) \times (X - E[X])] \\ &= 2Var[X] - 5Cov[W,X] \\ &= 2 \times 9 - 5 \times (-4) = 38. \end{aligned}$$

Problem 134 *If X is a random variable with outcomes $1, 2, 3$ and with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ then calculate $E[X]$, $E[X]^2$, $E\left[X^2\right]$, $Var[X]$, $E\left[e^X\right]$ and $e^{E[X]}$.*

Answer

We have:

$$\begin{aligned} E[X] &= 1 \times \frac{1}{4} + 2 \times \frac{1}{2} + 3 \times \frac{1}{4} = 2 \\ E[X]^2 &= 2^2 = 4 \\ E\left[X^2\right] &= 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{2} + 3^2 \times \frac{1}{4} = \frac{9}{2} \\ Var[X] &= E\left[X^2\right] - E[X]^2 = \frac{9}{2} - 2^2 = \frac{1}{2} \\ E\left[e^X\right] &= e^1 \times \frac{1}{4} + e^2 \times \frac{1}{2} + e^3 \times \frac{1}{4} = 9.3955 \\ e^{E[X]} &= e^2 = 7.3981. \end{aligned}$$

3.4 Econometrics

Problem 135 *Prove that*

$$S(m) = \sum_{i=1}^{n} (Y_i - m)^2$$

is minimized when $m = \hat{m} = \overline{X}$ *both with calculus and without calculus.*

Answer

Using the sum rule for derivatives we have:

$$\begin{aligned} \frac{dS(m)}{dm} &= \sum_{i=1}^{n} \frac{d}{dm} (Y_i - m)^2 \\ &= \sum_{i=1}^{n} -2 (Y_i - m) \end{aligned}$$

so that the first-order conditions are:

$$\begin{aligned} \frac{dS(\hat{m})}{dm} &= 0 = \sum_{i=1}^{n} -2 (Y_i - \hat{m}) \\ &\Longrightarrow -2 \sum_{i=1}^{n} (Y_i - \hat{m}) = 0 \\ &\Longrightarrow \sum_{i=1}^{n} Y_i - n\hat{m} = 0 \\ &\Longrightarrow \hat{m} = \frac{1}{n} \sum_{i=1}^{n} Y_i = \bar{Y} \end{aligned}$$

with:

$$\begin{aligned}\frac{d^2 S(m)}{dm^2} &= \sum_{i=1}^{n} \frac{d}{dm}\left(-2\left(Y_i - m\right)\right) \\ &= \sum_{i=1}^{n} 2 = 2n > 0\end{aligned}$$

so that $S(m)$ is globally convex so that $\bar{Y}$ is a global minimum.

Without using calculus we have (completing the square):

$$\begin{aligned}S(m) &= \sum_{i=1}^{n}(Y_i - m)^2 = \sum_{i=1}^{n}\left(\left(Y_i - \bar{Y}\right) + \left(m - \bar{Y}\right)\right)^2 \\ &= \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 + 2\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)\left(m - \bar{Y}\right) + \sum_{i=1}^{n}\left(m - \bar{Y}\right)^2 \\ &= \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 + 2\left(m - \bar{Y}\right)\underbrace{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)}_{=0} + n\left(m - \bar{Y}\right)^2 \\ &= \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 + n\left(m - \bar{Y}\right)^2.\end{aligned}$$

Note that the term $\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2$ does not depend on m while for the second term $n\left(m - \bar{Y}\right)^2 > 0$ for $m \neq \bar{Y}$ and $n\left(m - \bar{Y}\right)^2 = 0$ for $m = \bar{Y}$ so that $m = \bar{Y}$ is a global minimum of $S(m)$.

Problem 136 *Consider the simple linear regression model without a constant:*

$$Y_i = \beta X_i + e_i,\ i = 1, 2, \ldots n.$$

The least squares estimator $\hat{\beta}$ minimizes:

$$S(\beta) = \sum_{i=1}^{n}\left(Y_i - \beta X_i\right)^2.$$

Show from the first-order conditions that

$$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}$$

and that the second-order conditions for a minimum will be satisfied. Show that $\hat{\beta}$ is the least squares estimator without using calculus. Show that if $\hat{e}_i = Y_i - X_i\hat{\beta}$ is the least squares residual then:

$$\sum_{i=1}^{n} Y_i^2 = \hat{\beta}^2 \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \hat{e}_i^2.$$

Answer

Using the sum rule for derivatives we have:

$$\begin{aligned}\frac{dS(\beta)}{d\beta} &= \sum_{i=1}^{n}\frac{d}{d\beta}(Y_i - X_i\beta)^2\\ &= \sum_{i=1}^{n} -2X_i(Y_i - X_i\beta)\end{aligned}$$

so that the first-order conditions are:

$$\begin{aligned}\frac{dS\left(\hat{\beta}\right)}{d\beta} &= 0 = \sum_{i=1}^{n} -2X_i\left(Y_i - X_i\hat{\beta}\right)\\ &\Longrightarrow -2\sum_{i=1}^{n} X_i\left(Y_i - X_i\hat{\beta}\right) = 0\\ &\Longrightarrow \sum_{i=1}^{n} X_iY_i - \left(\sum_{i=1}^{n} X_i^2\right)\hat{\beta} = 0\\ &\Longrightarrow \hat{\beta} = \frac{\sum_{i=1}^{n} X_iY_i}{\sum_{i=1}^{n} X_i^2}\end{aligned}$$

with:

$$\begin{aligned}\frac{d^2S(\beta)}{d\beta^2} &= -2\sum_{i=1}^{n}\frac{d}{d\beta}X_i(Y_i - X_i\beta)\\ &= 2\sum_{i=1}^{n} X_i^2 > 0\end{aligned}$$

so that $S(\beta)$ is globally convex so that $\hat{\beta}$ is a global minimum.

Without using calculus we have (completing the square):

$$\begin{aligned}S(\beta) &= \sum_{i=1}^{n}(Y_i - X_i\beta)^2 = \sum_{i=1}^{n}\left(\left(Y_i - X_i\hat{\beta}\right) + X_i\left(\hat{\beta} - \beta\right)\right)^2\\ &= \sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2 + 2\sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)X_i\left(\hat{\beta} - \beta\right) + \sum_{i=1}^{n} X_i^2\left(\hat{\beta} - \beta\right)^2\\ &= \sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2 + 2\left(\hat{\beta} - \beta\right)\underbrace{\sum_{i=1}^{n} X_i\left(Y_i - X_i\hat{\beta}\right)}_{=0} + \left(\hat{\beta} - \beta\right)^2\sum_{i=1}^{n} X_i^2\\ &= \sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2 + \left(\hat{\beta} - \beta\right)^2\sum_{i=1}^{n} X_i^2.\end{aligned}$$

where the middle term is zero since:

$$\begin{aligned}\sum_{i=1}^{n} X_i\left(Y_i - X_i\hat{\beta}\right) &= \sum_{i=1}^{n} X_iY_i - \left(\sum_{i=1}^{n} X_i^2\right)\hat{\beta} \\ &= \sum_{i=1}^{n} X_iY_i - \left(\sum_{i=1}^{n} X_i^2\right)\frac{\sum_{i=1}^{n} X_iY_i}{\sum_{i=1}^{n} X_i^2} \\ &= \sum_{i=1}^{n} X_iY_i - \sum_{i=1}^{n} X_iY_i = 0.\end{aligned}$$

Now from

$$S(\beta) = \sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2 + \left(\hat{\beta} - \beta\right)^2\sum_{i=1}^{n} X_i^2$$

the term $\sum_{i=1}^{n}\left(Y_i - X_i\hat{\beta}\right)^2$ does not depend on β while the second term is positive for $\beta \neq \hat{\beta}$ and 0 for $\beta = \hat{\beta}$. It follows that $\hat{\beta}$ is a global minimum of $S(\beta)$.

Problem 137 *For the simple linear regression model:* $Y_i = \alpha + \beta X_i + e_i$, $i = 1, 2, \ldots n$ *the least squares estimators* $\hat{\alpha}$ *and* $\hat{\beta}$ *are the values of* α *and* β *which minimize the sum of squares function:*

$$S(\alpha, \beta) = \sum_{i=1}^{n}(Y_i - \alpha - \beta X_i)^2.$$

Show that:

$$\begin{aligned}\hat{\alpha} &= \bar{Y} - \hat{\beta}\bar{X} \\ \hat{\beta} &= \frac{n\sum_{i=1}^{n} X_iY_i - \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)}{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2} \\ &= \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}.\end{aligned}$$

Show that $S(\alpha, \beta)$ *is globally convex as long as the* X_i *'s are not all identical.*

Answer

We have using the sum and chain rules that:

$$\begin{aligned}\frac{\partial S(\alpha, \beta)}{\partial \alpha} &= -2\sum_{i=1}^{n}(Y_i - \alpha - \beta X_i) \\ \frac{\partial S(\alpha, \beta)}{\partial \beta} &= -2\sum_{i=1}^{n} X_i(Y_i - \alpha - \beta X_i)\end{aligned}$$

so that the first-order conditions for a minimum are:

$$\begin{aligned}\frac{\partial S\left(\hat{\alpha},\hat{\beta}\right)}{\partial\alpha} &= -2\sum_{i=1}^{n}\left(Y_i-\hat{\alpha}-\hat{\beta}X_i\right)=0\Longrightarrow n\hat{\alpha}+\left(\sum_{i=1}^{n}X_i\right)\hat{\beta}=\sum_{i=1}^{n}Y_i\\ \frac{\partial S\left(\hat{\alpha},\hat{\beta}\right)}{\partial\beta} &= -2\sum_{i=1}^{n}X_i\left(Y_i-\hat{\alpha}-\hat{\beta}X_i\right)=0\Longrightarrow\left(\sum_{i=1}^{n}X_i\right)\hat{\alpha}+\left(\sum_{i=1}^{n}X_i^2\right)\hat{\beta}=\sum_{i=1}^{n}X_iY_i.\end{aligned}$$

From the first first-order condition we have:

$$\begin{aligned}n\hat{\alpha}+\left(\sum_{i=1}^{n}X_i\right)\hat{\beta} &= \sum_{i=1}^{n}Y_i\Longrightarrow\hat{\alpha}+\bar{X}\hat{\beta}=\bar{Y}\\ &\Longrightarrow\ \hat{\alpha}=\bar{Y}-\bar{X}\hat{\beta}.\end{aligned}$$

Writing both first-order conditions in matrix notation we have:

$$\left[\begin{array}{cc} n & \sum_{i=1}^{n}X_i\\ \sum_{i=1}^{n}X_i & \sum_{i=1}^{n}X_i^2\end{array}\right]\left[\begin{array}{c}\hat{\alpha}\\ \hat{\beta}\end{array}\right]=\left[\begin{array}{c}\sum_{i=1}^{n}Y_i\\ \sum_{i=1}^{n}X_iY_i\end{array}\right].$$

Solving for $\hat{\beta}$ using Cramer's rule we find that:

$$\hat{\beta}=\frac{n\sum_{i=1}^{n}X_iY_i-\left(\sum_{i=1}^{n}X_i\right)\left(\sum_{i=1}^{n}Y_i\right)}{n\sum_{i=1}^{n}X_i^2-\left(\sum_{i=1}^{n}X_i\right)^2}.$$

Verify that:

$$\begin{aligned}n\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right) &= n\sum_{i=1}^{n}X_iY_i-\left(\sum_{i=1}^{n}X_i\right)\left(\sum_{i=1}^{n}Y_i\right)\\ n\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2 &= n\sum_{i=1}^{n}X_i^2-\left(\sum_{i=1}^{n}X_i\right)^2\end{aligned}$$

and so:

$$\hat{\beta}=\frac{n\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)}{n\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}=\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)}{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}.$$

The Hessian of $S(\alpha,\beta)$ is given by:

$$H=\left[\begin{array}{cc}2n & 2\sum_{i=1}^{n}X_i\\ 2\sum_{i=1}^{n}X_i & 2\sum_{i=1}^{n}X_i^2\end{array}\right].$$

Using leading principal minors we have:

$$\begin{aligned}M_1 &= 2n>0\\ M_2 &= \det[H]=4\left(n\sum_{i=1}^{n}X_i^2-\left(\sum_{i=1}^{n}X_i\right)^2\right)\\ &= 4n\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2>0\end{aligned}$$

as long as not all $X_i's$ are identical since in that case there will be at least one $X_i \neq \bar{X}$ and so $\left(X_i - \bar{X}\right)^2 > 0$. Thus $\hat{\alpha}, \hat{\beta}$ is a unique global minimum.

Problem 138 *Let $\hat{Y}_i = \hat{\alpha} + \hat{\beta} X_i$ be the fitted value and let $\hat{e}_i = Y_i - \hat{\alpha} - \hat{\beta} X_i = Y_i - \hat{Y}_i$ be the least squares residual. Show that:*

$$\begin{aligned} \sum_{i=1}^{n} \hat{e}_i &= 0, \ \bar{Y} = \hat{\alpha} + \hat{\beta}\bar{X} \\ \sum_{i=1}^{n} X_i \hat{e}_i &= 0, \sum_{i=1}^{n} \hat{Y}_i \hat{e}_i = 0. \end{aligned}$$

Use this to show that:

$$\begin{aligned} \sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2 &= \sum_{i=1}^{n} \left(\hat{Y}_i - \bar{Y}\right)^2 + \sum_{i=1}^{n} \hat{e}_i^2 \\ &= \hat{\beta}^2 \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 + \sum_{i=1}^{n} \hat{e}_i^2 \end{aligned}$$

and from this that $0 \leq R^2 \leq 1$ where:

$$R^2 = \frac{\sum_{i=1}^{n} \left(\hat{Y}_i - \bar{Y}\right)^2}{\sum_{i=1}^{n} \left(Y_i - \bar{Y}\right)^2}.$$

Answer

From the first-order conditions we have:

$$\begin{aligned} \frac{\partial S\left(\hat{\alpha}, \hat{\beta}\right)}{\partial \alpha} &= -2 \sum_{i=1}^{n} \overbrace{\left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right)}^{\hat{e}_i} = 0 \Longrightarrow \sum_{i=1}^{n} \hat{e}_i = 0 \\ &\Longrightarrow n\hat{\alpha} + \left(\sum_{i=1}^{n} X_i\right) \hat{\beta} = \sum_{i=1}^{n} Y_i \Longrightarrow \bar{Y} = \hat{\alpha} + \hat{\beta}\bar{X}, \\ \frac{\partial S\left(\hat{\alpha}, \hat{\beta}\right)}{\partial \beta} &= -2 \sum_{i=1}^{n} X_i \overbrace{\left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right)}^{\hat{e}_i} = 0 \Longrightarrow \sum_{i=1}^{n} X_i \hat{e}_i = 0. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=1}^{n} \hat{Y}_i \hat{e}_i &= \sum_{i=1}^{n} \left(\hat{\alpha} + \hat{\beta} X_i\right) \hat{e}_i \\ &= \hat{\alpha} \sum_{i=1}^{n} \hat{e}_i + \hat{\beta} \sum_{i=1}^{n} X_i \hat{e}_i = 0. \end{aligned}$$

Now:

$$\begin{aligned}
Y_i &= \hat{Y}_i + \hat{e}_i \Longrightarrow Y_i - \bar{Y} = \hat{Y}_i - \bar{Y} + \hat{e}_i \\
&\Longrightarrow \left(Y_i - \bar{Y}\right)^2 = \left(\hat{Y}_i - \bar{Y}\right)^2 + \hat{e}_i^2 + 2\left(\hat{Y}_i - \bar{Y}\right)\hat{e}_i \\
&\Longrightarrow \sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 = \sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2 + \sum_{i=1}^{n}\hat{e}_i^2 + 2\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)\hat{e}_i.
\end{aligned}$$

But:

$$\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)\hat{e}_i = \sum_{i=1}^{n}\hat{Y}_i\hat{e}_i - \bar{Y}\sum_{i=1}^{n}\hat{e}_i = 0$$

so that:

$$\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 = \sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2 + \sum_{i=1}^{n}\hat{e}_i^2.$$

Therefore dividing both sides by: $\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2$ we have:

$$1 = \frac{\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2 + \sum_{i=1}^{n}\hat{e}_i^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} = R^2 + \frac{\sum_{i=1}^{n}\hat{e}_i^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}$$

so that since: $\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2 \geq 0$ and $\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2 > 0$ we have:

$$R^2 = \frac{\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} \geq 0$$

and since $\sum_{i=1}^{n}\hat{e}_i^2 \geq 0$ we have:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}\hat{e}_i^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2} \leq 1.$$

Problem 139 *Now consider the* ***weighted*** *sum of squares function given by:*

$$S_W(\alpha, \beta) = \sum_{i=1}^{n} w_i \left(Y_i - \alpha - \beta X_i\right)^2$$

where the weights w_i satisfy: $\sum_{i=1}^{n} w_i = n$. Show that if $\hat{\beta}$ minimizes $S_W(\alpha, \beta)$, then:

$$\hat{\beta} = \frac{n\sum_{i=1}^{n} w_i X_i Y_i - \left(\sum_{i=1}^{n} w_i X_i\right)\left(\sum_{i=1}^{n} w_i Y_i\right)}{n\sum_{i=1}^{n} w_i X_i^2 - \left(\sum_{i=1}^{n} w_i X_i\right)^2}.$$

Answer

We have using the sum and chain rules that:

$$\begin{aligned}\frac{\partial S(\alpha,\beta)}{\partial \alpha} &= -2\sum_{i=1}^{n} w_i (Y_i - \alpha - \beta X_i)\\ \frac{\partial S(\alpha,\beta)}{\partial \beta} &= -2\sum_{i=1}^{n} w_i X_i (Y_i - \alpha - \beta X_i)\end{aligned}$$

so that the first-order conditions for a minimum are:

$$\begin{aligned}\frac{\partial S\left(\hat{\alpha},\hat{\beta}\right)}{\partial \alpha} &= -2\sum_{i=1}^{n} w_i \left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right) = 0 \Longrightarrow \overbrace{\left(\sum_{i=1}^{n} w_i\right)}^{=n}\hat{\alpha} + \left(\sum_{i=1}^{n} w_i X_i\right)\hat{\beta} = \sum_{i=1}^{n} w_i Y_i\\ \frac{\partial S\left(\hat{\alpha},\hat{\beta}\right)}{\partial \beta} &= -2\sum_{i=1}^{n} w_i X_i \left(Y_i - \hat{\alpha} - \hat{\beta} X_i\right) = 0 \Longrightarrow \left(\sum_{i=1}^{n} w_i X_i\right)\hat{\alpha} + \left(\sum_{i=1}^{n} w_i X_i^2\right)\hat{\beta} = \sum_{i=1}^{n} w_i X_i Y_i\end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} n & \sum_{i=1}^{n} w_i X_i \\ \sum_{i=1}^{n} w_i X_i & \sum_{i=1}^{n} w_i X_i^2 \end{bmatrix}\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{n} w_i Y_i \\ \sum_{i=1}^{n} w_i X_i Y_i\end{bmatrix}.$$

Solving for $\hat{\beta}$ using Cramer's rule we find that:

$$\hat{\beta} = \frac{n\sum_{i=1}^{n} w_i X_i Y_i - \left(\sum_{i=1}^{n} w_i X_i\right)\left(\sum_{i=1}^{n} w_i Y_i\right)}{n\sum_{i=1}^{n} w_i X_i^2 - \left(\sum_{i=1}^{n} w_i X_i\right)^2}.$$

The following 3 problems are based on the following information:

Let p_i $i = 1, 2, 3...m$ be the probability that an individual randomly chosen from the general population has an income in category i (say \$20,000/year to \$25,000/year) and where there are m income categories. Alternatively p_i is the proportion of the population in income class i. These probabilities describe the income distribution of society. You as an econometrician do not know these probabilities: $p_1, p_2, \ldots p_m$. However, you have surveyed n people taken randomly from the population. Let n_i be the number of people in your sample in income category i. Clearly $n_1 + n_2 + \cdots + n_m = n$. You would like to use these numbers to estimate the p_i $'s$ and you decide to do this by choosing $\hat{p}_i$ $i = 1, 2, ...m$ to maximize the log likelihood given by:

$$l(p_1, p_2, \ldots p_m) = \sum_{i=1}^{m} n_i \ln(p_i).$$

Of course you realize that probabilities sum to 1 so that the following constraint holds:

$$p_1 + p_2 + \cdots + p_m = 1.$$

Problem 140 *What is the Lagrangian for this constrained maximization problem? From the first order conditions show that the maximum likelihood estimates are given by:* $\hat{p}_i = \frac{n_i}{n}$*. Why is this sensible?*

Answer

We have:

$$\mathcal{L}(\lambda, p_1, p_2, \ldots p_m) = \sum_{i=1}^{m} n_i \ln(p_i) + \lambda \left(1 - \sum_{i=1}^{m} p_i\right)$$

so that the first-order conditions are:

$$\begin{aligned} 1 - \sum_{i=1}^{m} \hat{p}_i &= 0 \Longrightarrow 1 = \sum_{i=1}^{m} \hat{p}_i \\ \frac{n_i}{\hat{p}_i} - \hat{\lambda} &= 0 \text{ for } i = 1, 2, \ldots n \Longrightarrow n_i = \hat{\lambda}\hat{p}_i. \end{aligned}$$

It follows then that:

$$n_i = \hat{\lambda}\hat{p}_i \Longrightarrow n = \sum_{i=1}^{m} n_i = \sum_{i=1}^{m} \hat{\lambda}\hat{p}_i = \hat{\lambda} \sum_{i=1}^{m} \hat{p}_i = \hat{\lambda}$$

so that: $\hat{\lambda} = n$ and: $\hat{p}_i = \frac{n_i}{\hat{\lambda}} = \frac{n_i}{n}$.

Problem 141 *If* $l^*(n_1, n_2 \ldots\ldots n_m) = l(\hat{p}_1, \hat{p}_2 \ldots\ldots \hat{p}_m)$*, what is* $\frac{\partial l^*}{\partial n_i}$ *equal to?*

Answer

From the envelope theorem we have:

$$\begin{aligned} \frac{\partial l^*}{\partial n_i} &= \frac{\partial}{\partial n_i} \mathcal{L}(\lambda, p_1, p_2, \ldots p_m) \, |_{p_i = \hat{p}_i, \lambda = \hat{\lambda}} \\ &= \frac{\partial}{\partial n_i} \left(\sum_{i=1}^{m} n_i \ln(p_i) + \lambda \left(1 - \sum_{i=1}^{m} p_i\right)\right) |_{p_i = \hat{p}_i, \lambda = \hat{\lambda}} \\ &= \ln(p_i) \, |_{p_i = \hat{p}_i, \lambda = \hat{\lambda}} \\ &= \ln(\hat{p}_i). \end{aligned}$$

Problem 142 *An entropy measure of the inequality of income is given by:*

$$E(p_1, p_2, \ldots p_m) = -\sum_{i=1}^{m} p_i \ln(p_i)$$

where the greater is E *the greater is the income inequality. Show that the minimum of this measure of income inequality occurs when* $p_i = \frac{1}{m}$ *(remember the constraint that probabilities sum to one!) Show that the estimated income inequality satisfies:*

$$E(\hat{p}_1, \hat{p}_2, \ldots \hat{p}_m) = -l(\hat{p}_1, \hat{p}_2, \ldots \hat{p}_m)/n.$$

Answer

The Lagrangian is:

$$\mathcal{L}(\lambda, p_1, p_2, \ldots p_m) = -\sum_{i=1}^{m} p_i \ln(p_i) + \lambda \left(1 - \sum_{i=1}^{m} p_i\right)$$

with first-order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}\left(\hat{\lambda}, \hat{p}_1, \hat{p}_2, \ldots \hat{p}_m\right)}{\partial \lambda} &= 0 = 1 - \sum_{i=1}^{m} \hat{p}_i \\
\frac{\partial \mathcal{L}\left(\hat{\lambda}, \hat{p}_1, \hat{p}_2, \ldots \hat{p}_m\right)}{\partial p_i} &= 1 + \ln(\hat{p}_i) - \hat{\lambda} = 0 \text{ for } i = 1, 2, \ldots m.
\end{aligned}$$

It follows then that:

$$1 + \ln(\hat{p}_i) - \hat{\lambda} = 0 \Longrightarrow \hat{p}_i = e^{\hat{\lambda}-1}$$

and:

$$1 - \sum_{i=1}^{m} \hat{p}_i = 0 \Longrightarrow \sum_{i=1}^{m} e^{\hat{\lambda}-1} = 1 \Longrightarrow m e^{\hat{\lambda}-1} = 1 \Longrightarrow \hat{\lambda} = 1 - \ln(m)$$

and so:

$$\hat{p}_i = e^{\hat{\lambda}-1} = e^{-\ln(m)} = \frac{1}{m} \text{for } i = 1, 2, \ldots m.$$

The objective function is convex $-\sum_{i=1}^{m} p_i \ln(p_i)$ since the Hessian is:

$$H(p_1, p_2, \ldots p_m) = \begin{bmatrix} \frac{1}{p_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{p_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{p_m} \end{bmatrix}$$

and so is diagonal with positive elements along the diagonal and hence is positive definite. Since the constraint is linear it then follows that the solution to the first-order conditions is a global minimum.

Chapter 4

Dynamics

4.1 Complex Variables and Trigonometry

Problem 143 *Given the complex number:* $\frac{2}{3} - \frac{1}{2}i$ *calculate the absolute value:* $\left|\frac{2}{3} - \frac{1}{2}i\right|$ *and the conjugate:* $\overline{\frac{2}{3} - \frac{1}{2}i}$. *Will* $\left(\frac{2}{3} - \frac{1}{2}i\right)^t \to 0$ *as* $t \to \infty$*?*

Answer

We have:

$$\begin{aligned} \left|\frac{2}{3} - \frac{1}{2}i\right| &= \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{5}{6} \\ \overline{\frac{2}{3} - \frac{1}{2}i} &= \frac{2}{3} + \frac{1}{2}i. \end{aligned}$$

Since $\left|\frac{2}{3} - \frac{1}{2}i\right| = \frac{5}{6} < 1$ it follows that $\left(\frac{2}{3} - \frac{1}{2}i\right)^t \to 0$ as $t \to \infty$.

Problem 144 *What is:* $\left|e^{(a+bi)}\right|$ *and when will* $e^{(a+bi)t} \to 0$ *as* $t \to \infty$*? Explain. If* $a + bi = 7e^{3i}$*, then what are* a *and* b *equal to? If* $a + bi = e^{3+4i}$ *then what are* a *and* b *equal to?*

Answer

Since

$$|a + bi|^2 = a^2 + b^2 = (a + bi)(a - bi) = (a + bi)\left(\overline{a + bi}\right)$$

and since $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$ we have::

$$\begin{aligned} \overline{e^{(a+bi)}} &= \overline{\cos(a + bi) + i\sin(a + bi)} \\ &= \cos(a + bi) - i\sin(a + bi) \\ &= \cos(-(a + bi)) + i\sin(-(a + bi)) \\ &= e^{-(a+bi)t} \end{aligned}$$

and so we have:

$$\begin{aligned}\left|e^{(a+bi)}\right|^2 &= e^{(a+bi)}\overline{e^{(a+bi)}}\\ &= e^{(a+bi)}e^{(a-bi)}\\ &= e^{2a}\end{aligned}$$

so that:

$$\left|e^{(a+bi)}\right| = e^a.$$

Thus:

$$\left|e^{(a+bi)t}\right| = e^{at} \to 0$$

only if $a < 0$.

If $a + bi = 7e^{3i}$ then

$$\begin{aligned}a + bi &= 7e^{3i} = 7\left(\cos(3) + i\sin(3)\right)\\ &\Rightarrow a = 7\cos(3)\,,\ b = 7\sin(3)\,.\end{aligned}$$

If $a + bi = e^{3+4i}$ then

$$a + bi = e^{3+4i} = e^3\left(e^{4i}\right) = e^3\left(\cos(4) + i\sin(4)\right)$$

and so: $a = e^3\cos(4)$ and $b = e^3\sin(4)$.

Problem 145 *Show that:*

$$\cos(\theta) = \frac{e^{\theta i} + e^{-\theta i}}{2},\ \sin(\theta) = \frac{e^{\theta i} - e^{-\theta i}}{2i}.$$

Answer

Using $e^{\theta i} = \cos(\theta) + i\sin(\theta)$, $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$ we have:

$$\begin{aligned}\frac{e^{\theta i} + e^{-\theta i}}{2} &= \frac{\cos(\theta) + i\sin(\theta) + (\cos(-\theta) + i\sin(-\theta))}{2}\\ &= \frac{\cos(\theta) + i\sin(\theta) + (\cos(\theta) - i\sin(\theta))}{2}\\ &= \frac{2\cos(\theta)}{2} = \cos(\theta)\,.\end{aligned}$$

$$\begin{aligned}\frac{e^{\theta i} - e^{-\theta i}}{2i} &= \frac{\cos(\theta) + i\sin(\theta) - (\cos(-\theta) + i\sin(-\theta))}{2i}\\ &= \frac{\cos(\theta) + i\sin(\theta) - (\cos(\theta) - i\sin(\theta))}{2i}\\ &= \frac{2i\sin(\theta)}{2i} = \sin(\theta)\,.\end{aligned}$$

Problem 146 *Given the complex number* $0.8 - 0.7i$ *calculate the absolute value:* $|0.8 - 0.7i|$ *and the conjugate:* $\overline{0.8 - 0.7i}$. *Will* $(0.8 - 0.7i)^t \to 0$ *as* $t \to \infty$*?*

Answer

We have:

$$\begin{aligned} |0.8-0.7i| &= \sqrt{0.8^2+(-0.7)^2} = 1.063 \\ \overline{0.8-0.7i} &= 0.8+0.7i. \end{aligned}$$

Since $|0.8-0.7i|>1$ which violates the requirement $|a+bi|<1$ for $(a+bi)^t \to 0$, it follows that: $(0.8-0.7i)^t$ does not go to 0 as $t\to\infty$.

Problem 147 *Prove that:* $\left|e^{\theta i}\right| = 1.$

Answer

We have: $e^{\theta i} = \cos(\theta) + i\sin(\theta)$ and using the fact that: $\cos(\theta)^2 + \sin(\theta)^2 = 1$ that:

$$\begin{aligned} \left|e^{\theta i}\right| &= \sqrt{(\cos(\theta)+i\sin(\theta))(\cos(\theta)-i\sin(\theta))} \\ &= \sqrt{\cos(\theta)^2+\sin(\theta)^2} = \sqrt{1} = 1. \end{aligned}$$

Problem 148 *Consider the complex number:* $\frac{1}{2}+\frac{1}{2}i$. *Write this in polar form and calculate* $\left(\frac{1}{2}+\frac{1}{2}i\right)^t$ *for* $t=6$ *using the polar form. What will happen as* $t\to\infty$?

Answer

We have:

$$\frac{1}{2}+\frac{1}{2}i = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}} \Longrightarrow \left(\frac{1}{2}+\frac{1}{2}i\right)^t = \left(\frac{1}{\sqrt{2}}\right)^t e^{it\frac{\pi}{4}}$$

so that:

$$\begin{aligned} \left(\frac{1}{2}+\frac{1}{2}i\right)^6 &= \left(\frac{1}{\sqrt{2}}\right)^6 e^{i6\frac{\pi}{4}} \\ &= \frac{1}{8}\left(\cos\left(\frac{3}{2}\pi\right)+i\sin\left(\frac{3}{2}\pi\right)\right) = -\frac{1}{8}i. \end{aligned}$$

Since:

$$\left|\frac{1}{2}+\frac{1}{2}i\right| = \frac{1}{\sqrt{2}} = 0.707 < 1 \Longrightarrow \left(\frac{1}{2}+\frac{1}{2}i\right)^t \to 0.$$

Problem 149 *Find* $\ln\left(\frac{1}{2}+\frac{1}{2}i\right)$.

Answer

Since:

$$\begin{aligned}\frac{1}{2}+\frac{1}{2}i &= \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}} \Longrightarrow \ln\left(\frac{1}{2}+\frac{1}{2}i\right)=\ln\left(\frac{1}{\sqrt{2}}\right)+i\frac{\pi}{4} \\ &= -\frac{1}{2}\ln(2)+i\frac{\pi}{4}.\end{aligned}$$

4.2 Difference Equations

Problem 150 *Consider the difference equation:*

$$Y_t = 100+\frac{3}{2}Y_{t-1}-\frac{3}{4}Y_{t-2}$$

Find the equilibrium value Y^* and determine whether this difference equation is stable or not. If $Y_0 = Y_1 = 100$, calculate Y_t for $t = 2, 3, 4, 5$.

Answer

The equilibrium value Y^* is determined by:

$$Y^* = 100+\frac{3}{2}Y^*-\frac{3}{4}Y^* \Longrightarrow Y^* = 400.$$

The characteristic polynomial is:

$$r^2-\frac{3}{2}r+\frac{3}{4}=0$$

with complex roots:

$$r_1 = \frac{3}{4}+\frac{\sqrt{3}}{4}i,\ r_2 = \frac{3}{4}-\frac{\sqrt{3}}{4}i.$$

The difference equation is stable since:

$$|r_1| = |r_2| = \sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2} = \frac{\sqrt{3}}{2} = 0.866 < 1.$$

As well:

$$\begin{aligned}Y_2 &= 100+\frac{3}{2}\times 100-\frac{3}{4}\times 100 = 175 \\ Y_3 &= 100+\frac{3}{2}\times 175-\frac{3}{4}\times 100 = 287.5 \\ Y_4 &= 100+\frac{3}{2}\times 287.5-\frac{3}{4}\times 175 = 400 \\ Y_5 &= 100+\frac{3}{2}\times 400-\frac{3}{4}\times 287.5 = 484.38 \\ Y_6 &= 100+\frac{3}{2}\times 484.38-\frac{3}{4}\times 400 = 526.57.\end{aligned}$$

Problem 151 *Consider the difference equation:*

$$Y_t = 100 + 1.3Y_{t-1} - 0.4Y_{t-2}.$$

If $Y_0 = Y_1 = 100$, calculate Y_t for $t = 2, 3, 4$. What is the equilibrium value Y^*? If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, calculate r_1 and r_2. Use r_1 and r_2 to determine if this difference equation is stable or unstable. Calculate A_1 and A_2.

Answer

The equilibrium value Y^* is determined by:

$$Y^* = 100 + 1.3Y^* - 0.4Y^* \Longrightarrow Y^* = 1000.$$

The characteristic polynomial is:

$$r^2 - 1.3r + 0.4 = 0$$

with complex roots:

$$r_1 = 0.5, r_2 = 0.8.$$

The difference equation is stable since:

$$|r_1| = 0.5 < 1, \ |r_2| = 0.8 < 1.$$

As well:

$$\begin{aligned} Y_2 &= 100 + 1.3 \times 100 - 0.4 \times 100 = 190 \\ Y_3 &= 100 + 1.3 \times 190 - 0.4 \times 100 = 307 \\ Y_4 &= 100 + 1.3 \times 307 - 0.4 \times 190 = 423.1 \\ Y_5 &= 100 + 1.3 \times 423.1 - 0.4 \times 307 = 527.23 \\ Y_6 &= 100 + 1.3 \times 527.23 - 0.4 \times 423.1 = 616.16. \end{aligned}$$

To calculate A_1 and A_2 we have:

$$\begin{aligned} Y_0 &= 100 = 1000 + A_1 + A_2 \Longrightarrow A_1 + A_2 = -900 \\ Y_1 &= 100 = 1000 + A_1 r_1 + A_2 r_2 \Longrightarrow r_1 A_1 + r_2 A_2 = -900 \end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} 1 & 1 \\ 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -900 \\ -900 \end{bmatrix}$$

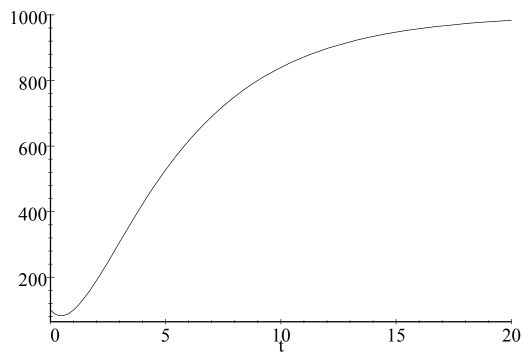
so that:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.5 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} -900 \\ -900 \end{bmatrix} = \begin{bmatrix} 600 \\ -1500 \end{bmatrix}$$

so the solution is:

$$Y_t = 1000 + 600 \times 0.5^t - 1500 \times 0.8^t$$

which is plotted below:



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Y_t

The following 2 problems are based on the following information: Consider the following version of the accelerator/multiplier model:

$$\begin{aligned} C_t &= 50 + \frac{3}{5} Y_{t-1} \\ I_t &= 50 + \frac{2}{5}\left(Y_{t-1} - Y_{t-2}\right) \\ G_t &= 50 \\ Y_t &= C_t + I_t + G_t \end{aligned}$$

Problem 152 *Show that Y_t follows a second order difference equation:*

$$Y_t = a_0 + a_1 Y_{t-1} + a_2 Y_{t-2}$$

and calculate a_o, a_1, and a_2. Find the equilibrium value of Y_t. Calculate Y_t for $t = 2, 3, 4, 5$ when $Y_0 = 200$ and $Y_1 = 200$

Answer

We have:

$$\begin{aligned} Y_t &= C_t + I_t + G_t = 50 + \frac{3}{5} Y_{t-1} + 50 + \frac{2}{5}\left(Y_{t-1} - Y_{t-2}\right) + 50 \\ &= 150 + Y_{t-1} - \frac{2}{5} Y_{t-2}. \end{aligned}$$

The equilibrium value comes from:

$$Y^* = 150 + Y^* - 0.4Y^* \Longrightarrow Y^* = \frac{150}{\frac{2}{5}} = 375.$$

As well:

$$\begin{aligned}
Y_2 &= 150 + 200 - \frac{2}{5} \times 200 = 270 \\
Y_3 &= 150 + 270 - \frac{2}{5} \times 200 = 340 \\
Y_4 &= 150 + 340 - \frac{2}{5} \times 270 = 382 \\
Y_5 &= 150 + 382 - \frac{2}{5} \times 340 = 396 \\
Y_6 &= 150 + 396 - \frac{2}{5} \times 382 = 393.2.
\end{aligned}$$

Problem 153 *Find the roots of the quadratic polynomial for this difference equation. Show that the economy is stable (note: the appropriate roots are complex numbers).*

Answer

We have:

$$r^2 - r + \frac{2}{5} = 0$$

so that:

$$r_1 = \frac{1}{2} + \frac{1}{10} i\sqrt{15},\ r_2 = \frac{1}{2} - \frac{1}{10} i\sqrt{15}$$

and:

$$|r_1| = |r_2| = \left| \frac{1}{2} + \frac{1}{10} i\sqrt{15} \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{15}}{10}\right)^2} = 0.632 < 1$$

so that the economy is stable.

Problem 154 *A second order difference equation*

$$Y_t = a_0 + a_1 Y_{t-1} + a_2 Y_{t-2}$$

can always be rewritten as a first-order vector difference equation as:

$$\begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Y_{t-2} \end{bmatrix} + \begin{bmatrix} a_0 \\ 0 \end{bmatrix}$$

or as $X_t = a + AX_{t-1}$ *where:*

$$X_t = \begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix},\ A = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix},\ a = \begin{bmatrix} a_0 \\ 0 \end{bmatrix}.$$

(This turns out to be a very useful idea. For example the Kalman filter, used in econometrics and also to put a man on the moon, is based on this idea!) Show that the equilibrium value of X_t is

$$X^* = (I - A)^{-1} a$$

and that:

$$X_t = X^* + A^t (X_t - X_0).$$

Show that if the eigenvalues of A be less than 1 in absolute value then $\tilde{Y}_t$ is stable. Show that the eigenvalues of A are the roots of the characteristic polynomial of Y_t.

Answer

If $X_t = X_{t-1} = X^*$ we have:

$$\begin{aligned} X_t &= a + AX_{t-1} \Rightarrow X^* = a + AX^* \\ &\Rightarrow X^* - AX^* = a \\ &\Rightarrow (I - A) X^* = a \\ &\Rightarrow X^* = (I - A)^{-1} a. \end{aligned}$$

Thus if $\tilde{X}_t = X_t - X^*$ we have:

$$X_t = a + AX_{t-1},\ X^* = a + AX^* \Rightarrow \tilde{X}_t = A\tilde{X}_{t-1}.$$

Thus:

$$\tilde{X}_t = A\tilde{X}_{t-1} \Rightarrow \tilde{X}_t = AA\tilde{X}_{t-2} = A^2\tilde{X}_{t-2} \Rightarrow \cdots \Rightarrow \tilde{X}_t = A^t\tilde{X}_0$$

or using $\tilde{X}_t = X_t - X^*$ we have:

$$X_t = X^* + A^t (X_t - X_0).$$

Since

$$A = C\Lambda C^{-1}$$

where Λ is a diagonal matrix with the eigenvalues along the diagonal so that:

$$A^t = C\Lambda^t C^{-1}$$

and hence $A^t \to 0$ iff $\Lambda^t \to 0$ which requires all eigenvalues be less than 1 in absolute value. The eigenvalues of A are given by the roots of :

$$\begin{aligned} \det \left[\begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] &= \det \begin{bmatrix} a_1 - \lambda & a_2 \\ 1 & -\lambda \end{bmatrix} \\ &= \lambda^2 - a_1\lambda - a_2 \end{aligned}$$

which is the characteristic polynomial of Y_t.

Problem 155 *Consider the difference equation which we considered earlier:*

$$Y_t = 100 + 1.3Y_{t-1} - 0.4Y_{t-2}.$$

where $Y_0 = Y_1 = 100$. *Write this in the form* $X_t = a + AX_{t-1}$ *considered in the previous question and show that* Y_t *is stable by calculating the eigenvalues of* A. *Write down the solution* $X_t = X^* + A^t (X_t - X_0)$ *for this model.*

Answer

Here :

$$X_t = \begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix}, \; A = \begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}, \; a = \begin{bmatrix} 100 \\ 0 \end{bmatrix}.$$

The eigenvalues of A are the roots of :

$$\begin{aligned} \det\left[\begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] &= \det\begin{bmatrix} a_1 - \lambda & a_2 \\ 1 & -\lambda \end{bmatrix} \\ &= \lambda^2 - 1.3\lambda + 0.4 = 0 \end{aligned}$$

and so:

$$\lambda_1 = 0.8, \; \lambda_2 = 0.5.$$

Note that these are identical to the roots of the characteristic polynomial (which is also the same) of:

$$r^2 - 1.3r + 0.4 = 0.$$

Since both eigenvalues are less than 1 in absolute value Y_t is stable. Alternatively $A^t = C\Lambda^t C^{-1}$ as (the computer calculates this):

$$A^t = \begin{bmatrix} 2.6667 & -1.6667 \\ 3.3333 & -3.3333 \end{bmatrix} \begin{bmatrix} 0.8^t & 0 \\ 0 & 0.5^t \end{bmatrix} \begin{bmatrix} 1.0 & -0.5 \\ 1.0 & -0.8 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since:

$$\Lambda^t = \begin{bmatrix} 0.8^t & 0 \\ 0 & 0.5^t \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The equilibrium value $X^* = (I - A)^{-1} a$ is

$$X^* = \begin{bmatrix} Y^* \\ Y^* \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$$

and so the equilibrium value is $Y^* = 1000$, as we calculated earlier.

The starting values are $Y_0 = Y_1 = 100$ or:

$$X_0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

so that $X_t = X^* + A^t (X_t - X_0)$ becomes:

$$\begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} + \begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}^t \left(\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} - \begin{bmatrix} 100 \\ 100 \end{bmatrix} \right)$$

or

$$\begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} + \begin{bmatrix} 1.3 & -0.4 \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 900 \\ 900 \end{bmatrix}$$

or replacing A^t by $C\Lambda^t C^{-1}$ given above:

$$\begin{aligned} \begin{bmatrix} Y_t \\ Y_{t-1} \end{bmatrix} &= \begin{bmatrix} 1000 \\ 1000 \end{bmatrix} + \begin{bmatrix} 2.6667 & -1.6667 \\ 3.3333 & -3.3333 \end{bmatrix} \begin{bmatrix} .8^t & 0 \\ 0 & .5^t \end{bmatrix} \begin{bmatrix} 1.0 & -.5 \\ 1.0 & -.8 \end{bmatrix} \begin{bmatrix} 900 \\ 900 \end{bmatrix} \\ &= \begin{bmatrix} 1000 + 1200.0 \times .8^t - 300.01 \times .5^t \\ 1000 + 1500.0 \times .8^t - 599.99 \times .5^t \end{bmatrix}. \end{aligned}$$

Problem 156 *Consider the difference equation:*

$$Y_t = 100 + Y_{t-1} - \frac{1}{2} Y_{t-2}$$

If $Y_0 = Y_1 = 100$, calculate Y_t for $t = 2, 3, 4, 5, 6$. If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, , calculate Y^* and explain what it is. Calculate r_1 and r_2. Determine if this difference equation is stable by calculating $| r_1 |$ and $| r_2 |$. What does the fact that r_1 and r_2 are complex tell you, or if they are not complex, what does that tell you? Calculate A_1 and A_2. Show that the solution can be written as:

$$Y_t = 200 - 200 \left(\frac{1}{\sqrt{2}} \right)^{t+1} \sin \left(\frac{\pi}{4} (t+1) \right).$$

Answer

The equilibrium value Y^* is determined by:

$$Y^* = 100 + Y^* - \frac{1}{2} Y^* \Longrightarrow Y^* = 200.$$

The characteristic polynomial is:

$$r^2 - r + \frac{1}{2} = 0$$

with complex roots:

$$r_1 = \frac{1}{2} + \frac{1}{2} i, r_2 = \frac{1}{2} - \frac{1}{2} i.$$

The difference equation is stable since:

$$|r_1| = |r_2| = \sqrt{\left(\frac{1}{2} \right)^2 + \left(\frac{1}{2} \right)^2} = \frac{\sqrt{2}}{2} = 0.707 < 1.$$

As well:

$$\begin{aligned}
Y_2 &= 100 + 100 - \frac{1}{2} \times 100 = 150 \\
Y_3 &= 100 + 150 - \frac{1}{2} \times 100 = 200 \\
Y_4 &= 100 + 200 - \frac{1}{2} \times 150 = 225 \\
Y_5 &= 100 + 225 - \frac{1}{2} \times 200 = 225 \\
Y_6 &= 100 + 225 - \frac{1}{2} \times 225 = 212.5.
\end{aligned}$$

To calculate A_1 and A_2 we have:

$$\begin{aligned}
Y_0 &= 100 = 200 + A_1 + A_2 \Longrightarrow A_1 + A_2 = -100 \\
Y_1 &= 100 = 200 + A_1 r_1 + A_2 r_2 \Longrightarrow r_1 A_1 + r_2 A_2 = -100
\end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{2} + \frac{1}{2}i & \frac{1}{2} - \frac{1}{2}i \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -100 \\ -100 \end{bmatrix}$$

so that:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} + \frac{1}{2}i & \frac{1}{2} - \frac{1}{2}i \end{bmatrix}^{-1} \begin{bmatrix} -100 \\ -100 \end{bmatrix} = \begin{bmatrix} -50 + 50i \\ -50 - 50i \end{bmatrix}.$$

Thus:

$$\begin{aligned}
Y_t &= 200 + (-50 + 50i)\left(\frac{1}{2} + \frac{1}{2}i\right)^t + (-50 - 50i)\left(\frac{1}{2} - \frac{1}{2}i\right)^t \\
&= 200 - 100\left(\left(\frac{1}{2} - \frac{1}{2}i\right)\left(\frac{1}{2} + \frac{1}{2}i\right)^t + \left(\frac{1}{2} + \frac{1}{2}i\right)\left(\frac{1}{2} - \frac{1}{2}i\right)^t\right) \\
&= 200 - 100\left(\frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}} \times \left(\frac{1}{\sqrt{2}}\right)^t e^{i\frac{\pi}{4}t} + \frac{1}{\sqrt{2}} e^{i\frac{\pi}{4}} \times \left(\frac{1}{\sqrt{2}}\right)^t e^{-i\frac{\pi}{4}t}\right) \\
&= 200 - 200\left(\frac{1}{\sqrt{2}}\right)^{t+1}\left(\frac{e^{i\frac{\pi}{4}(t+1)} + e^{-i\frac{\pi}{4}(t+1)}}{2}\right) \\
&= 200 - 200\left(\frac{1}{\sqrt{2}}\right)^{t+1} \sin\left(\frac{\pi}{4}(t+1)\right).
\end{aligned}$$

Problem 157 *Consider the difference equation:*

$$Y_t = 10 + 0.3Y_{t-1} + 0.4Y_{t-2}$$

If $Y_0 = Y_1 = 100$, calculate Y_t for $t = 2, 3, 4$. What is the equilibrium value Y^*? If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, calculate r_1 and r_2. Use r_1 and r_2 to determine if this difference equation is stable or unstable.

Answer

The equilibrium value Y^* is determined by:

$$Y^* = 100 + 0.3Y^* + 0.4Y^* \Longrightarrow Y^* = \frac{100}{0.3} = 333.33.$$

The characteristic polynomial is:

$$r^2 - 0.3r - 0.4 = 0$$

with roots:

$$r_1 = -0.5, r_2 = 0.8.$$

The difference equation is stable since:

$$|r_1| = 0.5 < 1,\ |r_2| = 0.8 < 1.$$

As well:

$$\begin{aligned}
Y_2 &= 100 + 0.3 \times 100 + 0.4 \times 100 = 170 \\
Y_3 &= 100 + 0.3 \times 170 + 0.4 \times 100 = 191 \\
Y_4 &= 100 + 0.3 \times 191 + 0.4 \times 170 = 225.3 \\
Y_5 &= 100 + 0.3 \times 243.99 + 0.4 \times 225.3 = 263.32 \\
Y_6 &= 100 + 0.3 \times 263.32 + 0.4 \times 225.3 = 269.12.
\end{aligned}$$

To calculate A_1 and A_2 we have:

$$\begin{aligned}
Y_0 &= 100 = 333.33 + A_1 + A_2 \Longrightarrow A_1 + A_2 = -233.33 \\
Y_1 &= 100 = 333.33 + A_1 r_1 + A_2 r_2 \Longrightarrow r_1 A_1 + r_2 A_2 = -233.33
\end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} 1 & 1 \\ -0.5 & 0.8 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -233.33 \\ -233.33 \end{bmatrix}$$

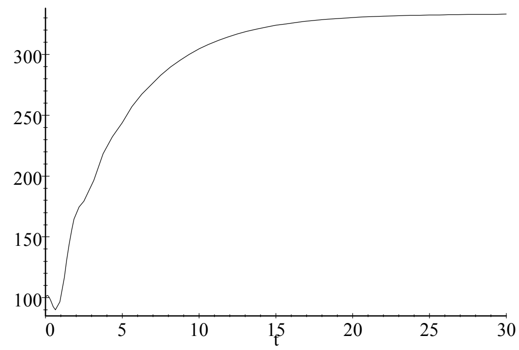
so that:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -0.5 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} -233.33 \\ -233.33 \end{bmatrix} = \begin{bmatrix} 35.897 \\ -269.23 \end{bmatrix}$$

so the solution is:

$$Y_t = 333.33 + 35.897 \times (-0.5)^t - 269.23 \times (0.8)^t$$

which is plotted below:



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Problem 158 *Consider the second order difference equation:*

$$\begin{aligned} Y_t &= 50 + 1.2Y_{t-1} - 0.3Y_{t-2} \\ Y_1 &= Y_o = 400. \end{aligned}$$

Calculate Y_t for $t = 2, 3, 4$. What is the equilibrium value Y^*? If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, calculate r_1 and r_2. Use r_1 and r_2 to determine if this difference equation is stable or unstable.

Answer

The equilibrium value Y^* is then:

$$Y^* = \frac{50}{1 - (1.2) - (-0.3)} = 500.$$

The characteristic polynomial is:

$$r^2 - 1.2r + 0.3 = 0$$

so that:

$$r_1, r_2 = \frac{-(-1.2) \pm \sqrt{(-1.2)^2 - 4 \times 0.3}}{2}$$

or:

$$r_1 = 0.355, r_2 = 0.845.$$

Since $| r_1 | < 1$ and $| r_2 | < 1$ this difference equation is stable; that is no matter what the starting values Y_t will converge to the equilibrium value $Y^* = 500$.

Problem 159 *Consider the second order difference equation:*

$$\begin{aligned} Y_t &= -50 + 0.6Y_{t-1} + 0.5Y_{t-2} \\ Y_1 &= 450, Y_o = 400. \end{aligned}$$

What is the equilibrium value Y^*? If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, calculate r_1 and r_2. Use r_1 and r_2 to determine if this difference equation is stable or unstable.

Answer

The equilibrium value Y^* is then:

$$Y^* = \frac{-50}{1 - (0.6) - (0.5)} = 500.$$

The characteristic polynomial is:

$$r^2 - 0.6r - 0.5 = 0$$

so that

$$r_1 = 1.0681, r_2 = -0.46811.$$

Since $r_1 = 1.07 > 1$ it follows that the difference equation is unstable.

Problem 160 *Consider the second order difference equation:*

$$\begin{aligned} Y_t &= 50 + 1.1Y_{t-1} - 0.6Y_{t-2} \\ Y_1 &= 50, Y_o = 60. \end{aligned}$$

Calculate Y_t for $t = 2, 3, 4$. What is the equilibrium value Y^*? If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, calculate r_1 and r_2. Use r_1 and r_2 to determine if this difference equation is stable or unstable.

Answer

The equilibrium value Y^* is then:

$$Y^* = \frac{50}{1 - (1.1) - (-0.6)} = 100.$$

The characteristic polynomial is:

$$r^2 - 1.1r + 0.6 = 0$$

so that:

$$r_1, r_2 = \frac{1 \pm \sqrt{(-1)^2 - 4 \times \frac{1}{2}}}{2}$$

or:

$$r_1 = 0.55 - 0.54544i,\ r_2 = 0.55 + 0.54544i.$$

Since

$$\mid r_1 \mid = \mid r_2 \mid = \sqrt{(0.55)^2 + (0.54544)^2} = 0.7746 < 1$$

the difference equation is stable.

Problem 161 *Consider the second order difference equation:*

$$\begin{aligned} Y_t &= 50 + 1.8Y_{t-1} - 1.3Y_{t-2} \\ Y_1 &= 100, Y_o = 120. \end{aligned}$$

/If $Y_0 = Y_1 = 100$, calculate Y_t for $t = 2, 3, 4$. What is the equilibrium value Y^*? If the solution is written as: $Y_t = Y^* + A_1 r_1^t + A_2 r_2^t$, calculate r_1 and r_2. Use r_1 and r_2 to determine if this difference equation is stable or unstable.

Answer

The equilibrium value Y^* is then:

$$Y^* = \frac{50}{1 - (1.8) - (-1.3)} = 100.$$

The characteristic polynomial is:

$$r^2 - 1.8r + 1.3 = 0$$

so that:

$$r_1 = 0.9 + 0.7i,\ r_2 = 0.9 - 0.7i.$$

Since

$$\mid r_1 \mid = \mid r_2 \mid = \sqrt{0.9^2 + 0.7^2} = 1.14 > 1$$

the difference equation is unstable.

Problem 162 *Suppose that Y_t is GNP, C_t is consumption, I_t is investment and G_t is government expenditure and that:*

$$\begin{aligned} C_t &= 100 + 0.6Y_{t-1} \\ I_t &= 50 + 0.5\left(Y_{t-1} - Y_{t-2}\right), \\ G_t &= 50 \\ Y_t &= C_t + I_t + G_t. \end{aligned}$$

Find the difference equation that Y_t follows, its equilibrium value Y^* and the solution to the difference equation. Is the economy stable?

Answer

It now follows that:

$$\begin{aligned} Y_t &= C_t + I_t + G_t \\ &= 100 + 0.6Y_{t-1} + 50 + 0.5\left(Y_{t-1} - Y_{t-2}\right) + 50 \end{aligned}$$

or

$$Y_t = 200 + 1.1Y_{t-1} - 0.5Y_{t-2}$$

which is a second-order difference equation.

The equilibrium value of GNP is thus:

$$Y^* = \frac{200}{1 - 1.1 - (-0.5)} = 500.$$

The characteristic polynomial is then:

$$r^2 - 1.1r + 0.5 = 0$$

with roots:

$$r_1 = 0.55 + 0.44i,\ r_2 = 0.55 - 0.44i.$$

Since:

$$|r_1| = |r_2| = \sqrt{0.55^2 + 0.44^2} = 0.704$$

the difference equation is stable.

4.3 Differential Equations

Problem 163 *Consider the differential equation:*

$$y''(t) + 5y'(t) + 4y(t) = 36.$$

Calculate the equilibrium value y^*. If $y(t) = y^* + A_1e^{r_1t} + A_2e^{r_2t}$, calculate r_1 and r_2 and from these determine whether or not this differential equation is stable. If $A_1 = A_2 = 1$ calculate:

$$\int_0^\infty \left(y(t) - y^*\right)^2 dt\ .$$

If $y(0) = 20$ and $y'(0) = -1$ find A_1 and A_2 and write down the solution $y(t)$.

Answer

The equilibrium value y^* is given by:

$$y^* = \frac{36}{4} = 9.$$

The characteristic polynomial is then:

$$\begin{aligned} r^2 + 5r + 4 &= 0 \\ &\implies r_1 = -4 < 0 \\ r_2 &= -1 < 0. \end{aligned}$$

Since $r_1 < 0$ and $r_2 < 0$ the differential equation is stable.

We have:

$$\begin{aligned} \int_0^\infty (y(t) - y^*)^2 \, dt &= \int_0^\infty \left(e^{-t} + e^{-4t}\right)^2 dt \\ &= \int_0^\infty \left(e^{-2t} + 2e^{-5t} + e^{-8t}\right) dt \\ &= \int_0^\infty e^{-2t} dt + 2\int_0^\infty e^{-5t} dt + \int_0^\infty e^{-8t} dt \\ &= \frac{1}{2} + 2 \times \frac{1}{5} + \frac{1}{8} = \frac{41}{40}. \end{aligned}$$

We have

$$\begin{aligned} 20 &= 9 + A_1 + A_2 \\ -1 &= -1 \times A_1 + -4 \times A_2 \end{aligned}$$

so that in matrix notation:

$$\begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

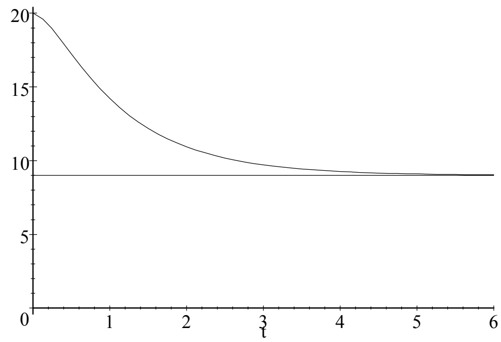
so that:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{43}{3} \\ -\frac{10}{3} \end{bmatrix}$$

and so the solution is:

$$y(t) = 9 + \frac{43}{3} e^{-t} - \frac{10}{3} e^{-4t}$$

which is plotted below:



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$Y(t)$

:

Problem 164 *Consider the differential equation:*

$$y''(t) + 3y'(t) + y(t) = 4, y(0) = 2, y'(0) = 1$$

If $y(t) = y^* + A_1 e^{r_1 t} + A_2 e^{r_2 t}$, *calculate* y^* *and explain what it is. Calculate* r_1 *and* r_2 *and determine whether or not this differential equation is stable. Calculate* A_1 *and* A_2.

Answer

The equilibrium value y^* is given by:

$$y^* = 4.$$

The characteristic polynomial is then:

$$\begin{aligned} r^2 + 3r + 1 &= 0 \\ &\Longrightarrow r_1 = -\frac{3}{2} + \frac{1}{2}\sqrt{5} = -0.38197 < 0 \\ r_2 &= -\frac{3}{2} - \frac{1}{2}\sqrt{5} = -2.618 < 0. \end{aligned}$$

Since $r_1 < 0$ and $r_2 < 0$ the differential equation is stable.

We have:

$$\begin{aligned} y(0) &= 2 = 4 + A_1 e^{r_1 0} + A_2 e^{r_2 0} \Longrightarrow A_1 + A_2 = -2 \\ y'(0) &= 1 = r_1 A_1 e^{r_1 0} + r_2 A_2 e^{r_2 0} \Longrightarrow r_1 A_1 + r_2 A_2 = 1 \end{aligned}$$

so that in matrix notation:

$$\begin{bmatrix} 1 & 1 \\ -\frac{3}{2} + \frac{1}{2}\sqrt{5} & -\frac{3}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

so that:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{3}{2}+\frac{1}{2}\sqrt{5} & -\frac{3}{2}-\frac{1}{2}\sqrt{5} \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.8944 \\ -0.10557 \end{bmatrix}.$$

Problem 165 *Consider:*

$$\begin{aligned} y''(t) + 8y'(t) + 2y(t) &= 6 \\ y(0) &= 2 \\ y'(0) &= 3 \end{aligned}$$

If $y(t) = y^* + A_1 e^{r_1 t} + A_2 e^{r_2 t}$, calculate y^* and explain what it is. Calculate r_1 and r_2 and determine whether or not this differential equation is stable. Calculate A_1 and A_2.

Answer

We have:

$$y^* = \frac{6}{2} = 3.$$

and the quadratic that defines r_1 and r_2 is:

$$r^2 + 8r + 2 = 0$$

with solutions:

$$\begin{aligned} r_1 &= -4 + \sqrt{14} = -0.25834 < 0 \\ r_2 &= -4 - \sqrt{14} = -7.7417 < 0. \end{aligned}$$

Since $r_1 < 0$ and $r_2 < 0$ we conclude that $y(t)$ stable.

To solve for A_1 and A_2 note that from:

$$y(t) = 3 + A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

it follows that:

$$\begin{bmatrix} 1 & 1 \\ -4+\sqrt{14} & -4-\sqrt{14} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

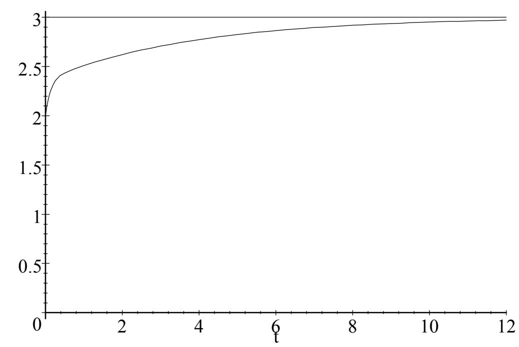
so:

$$\begin{aligned} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -4+\sqrt{14} & -4-\sqrt{14} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{28}\sqrt{14} - \frac{1}{2} \\ \frac{1}{28}\sqrt{14} - \frac{1}{2} \end{bmatrix}. \end{aligned}$$

The solution is therefore:

$$y(t) = 3 + \left(-\frac{1}{2} - \frac{1}{28}\sqrt{14}\right) e^{\left(-4+\sqrt{14}\right)t} - \frac{1}{28}\left(-1+\sqrt{14}\right)\sqrt{14}e^{-\left(4+\sqrt{14}\right)t}$$

which is plotted below



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$Y(t)$

Problem 166 *Consider:*

$$\begin{aligned} y''(t) - 3y'(t) - 2y(t) &= -4 \\ y(0) &= 1 \\ y'(0) &= 2. \end{aligned}$$

If $y(t) = y^* + A_1 e^{r_1 t} + A_2 e^{r_2 t}$, *calculate* y^* *and explain what it is. Calculate* r_1 *and* r_2 *and determine whether or not this differential equation is stable. Calculate* A_1 *and* A_2.

Answer

We have:

$$y^* = \frac{-4}{-2} = 2$$

and the quadratic that defines r_1 and r_2 is:

$$r^2 - 3r - 2 = 0$$

with solutions:

$$\begin{aligned} r_1 &= \frac{3}{2} + \frac{1}{2}\sqrt{17} > 0 \\ r_2 &= \frac{3}{2} - \frac{1}{2}\sqrt{17} < 0. \end{aligned}$$

Since $r_1 > 0$ it follows that (even though $r_2 < 0$) that the differential equation is unstable.

To solve for A_1 and A_2 note that from:

$$y(t) = 2 + A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

it follows that:

$$\begin{bmatrix} 1 & 1 \\ \frac{3}{2} + \frac{1}{2}\sqrt{17} & \frac{3}{2} - \frac{1}{2}\sqrt{17} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

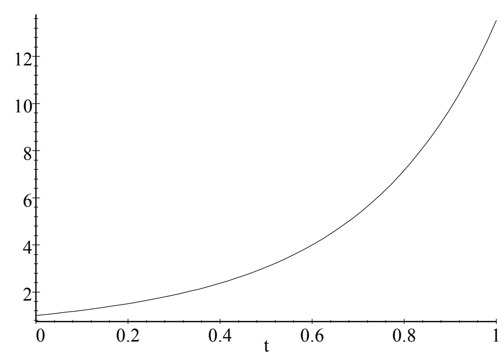
so:

$$\begin{aligned} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \frac{3}{2} + \frac{1}{2}\sqrt{17} & \frac{3}{2} - \frac{1}{2}\sqrt{17} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{34}\sqrt{17} - \frac{1}{2} \\ -\frac{7}{34}\sqrt{17} - \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Thus:

$$y(t) = 2 - 1.3489 \exp\left(-\frac{1}{2}\left(-3 + \sqrt{17}\right)t\right) + .34887 e^{\frac{1}{2}(3+\sqrt{17})t}$$

which is plotted below:



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Problem 167 *Consider:*

$$\begin{aligned} y''(t) + 3y'(t) + 5y(t) &= 10 \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

If $y(t) = y^* + A_1 e^{r_1 t} + A_2 e^{r_2 t}$*, calculate* y^* *and explain what it is. Calculate* r_1 *and* r_2 *and determine whether or not this differential equation is stable.*

Answer

We have:

$$y^* = \frac{10}{5} = 2$$

and the quadratic that defines r_1 and r_2 is:

$$r^2 + 3r + 5 = 0$$

so that:

$$\begin{aligned} r_1 &= -\frac{3}{2} + \frac{\sqrt{11}}{2} i \\ r_2 &= -\frac{3}{2} - \frac{\sqrt{11}}{2} i. \end{aligned}$$

Since the real part of r_1 and r_2, here $-\frac{3}{2}$, is negative, it follows that the differential equation is stable.

Problem 168 *Consider:*

$$\begin{aligned} y''(t) - 3y'(t) + 5y(t) &= 10 \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

If $y(t) = y^* + A_1 e^{r_1 t} + A_2 e^{r_2 t}$, calculate y^* and explain what it is. Calculate r_1 and r_2 and determine whether or not this differential equation is stable. Find A_1 and A_2 and the solution to $y(t)$.

Answer

We have:

$$y^* = \frac{10}{5} = 2$$

and the quadratic that defines r_1 and r_2 is:

$$r^2 - 3r + 5 = 0$$

so that:

$$\begin{aligned} r_1 &= \frac{3}{2} + \frac{\sqrt{11}}{2} i \\ r_2 &= \frac{3}{2} - \frac{\sqrt{11}}{2} i. \end{aligned}$$

Since the real part of r_1 and r_2, here $\frac{3}{2}$, is positive, it follows that the differential equation is unstable.

It follows that:

$$\begin{bmatrix} 1 & 1 \\ \frac{3}{2}+\frac{\sqrt{11}}{2}i & \frac{3}{2}-\frac{\sqrt{11}}{2}i \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

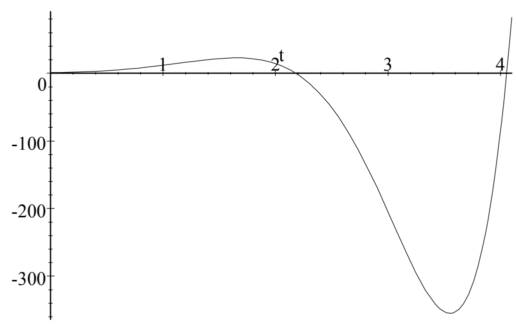
so:

$$\begin{aligned} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \frac{3}{2}+\frac{\sqrt{11}}{2}i & \frac{3}{2}-\frac{\sqrt{11}}{2}i \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}-\frac{1}{22}i\sqrt{11} \\ \frac{1}{2}+\frac{1}{22}i\sqrt{11} \end{bmatrix}. \end{aligned}$$

The exact solution is:

$$\begin{aligned} y(t) &= 2+\left(\frac{1}{2}-\frac{1}{22}i\sqrt{11}\right)e^{\left(\frac{3}{2}+\frac{\sqrt{11}}{2}i\right)t}+\left(\frac{1}{2}+\frac{1}{22}i\sqrt{11}\right)e^{\left(\frac{3}{2}-\frac{\sqrt{11}}{2}i\right)t} \\ &= 2+\frac{7}{11}\sqrt{11}e^{\frac{3}{2}t}\sin\left(\frac{1}{2}\sqrt{11}t\right)-e^{\frac{3}{2}t}\cos\left(\frac{1}{2}\sqrt{11}t\right) \end{aligned}$$

which is plotted below:



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