Economics Simplified

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Abstract

This paper argues that economics can be greatly simplified at no scientific cost. The cGE model, a GE model with Cobb-Douglas functional forms, is shown to be consistent with any conceivable economic data set. While GE and cGE models explain the data equally well, the cGE model is better in that it is simpler, it is linear, it has a unique equilibrium in $\mathbb{Q}$ instead of $\mathbb{R}$, and computation is easier. JEL A20, B40, C22, D58, E20.

1 Introduction

> *Thus the data will never be able to reveal backward bending indifference curves or non-convex indifference sets. Evidently, if we only have a finite number of observations, we will not be able to distinguish the case where the decision-maker's preferences are representable by a concave utility function from the case where his utility function is merely quasi-concave.*-Diewert (1973)

Every year economics becomes more complicated without, it seems, making better predictions. Being complicated has its costs: much of a professional economist's education is devoted to subtleties of real analysis, it's harder to understand why a model fails or succeeds, and it prevents scrutiny from outsiders.

In this paper we propose to simplify economics without affecting its predictive power. We apply Occam's razor to the core theory of economics: general equilibrium. What remains is what we call the cGE model: a general equilibrium (GE) model made up entirely of Cobb-Douglas (hence the c) functional forms. The cGE model is simple enough to be accessible to an undergraduate with calculus and linear algebra; general enough to be consistent with any conceivable data set.

Whether this simplification is desirable is a matter of perspective. From a mathematical perspective the *cGE* model is less general and hence inferior. If one views constructing complicated models as the most efficient means of allocating scarce academic resources, then being accessible to undergraduates makes the *cGE* model decidedly inferior. But from a scientific perspective the *cGE* model is both simpler and able to explain the data equally well, and hence by Occam's razor is *superior*. The *cGE* model is also superior from a pedagogical perspective by allowing economics undergraduates to understand general equilibrium without getting bogged down in the technicalities of real analysis. In this paper we assume simplification is desirable.

The data set we assume is idealized, better than any real-world data set, and shares two real-world characteristics: 1) the number of observations is finite and 2) the precision is finite (the data are in $\mathbb{Q}$). The data consists of all endowments and consumption bundles for all households, the prices of all goods and factors of production, and the production levels and factor demands for all industries. It is for *all* locations, time periods, and states of the world. States of the world could include either real or hypothetical policy experiments, and so the idealized data set would include the results of these policy experiments.

The main result of the paper is that there exists a *cGE* model that exactly reproduces this idealized data set. This has a number of implications.

First Cobb-Douglas is not testable. Far from being an extreme special case easily falsified by the data, the Cobb-Douglas functional form is flexible enough to fit any conceivable data set. Since the idealized data set can include policy experiments, it is also the case there is no policy experiment that can refute Cobb-Douglas.

Second, since *GE* includes *cGE*, the *GE* hypothesis is not testable. Since the idealized data set includes household endowments, this would seem to contradict Brown and Matzkin's (1996) result that the general equilibrium hypothesis is testable with endowment data. We show that Brown and Matzkin are in fact testing a *restricted* version of *GE*, and not *GE* itself.

Third, any generalization of *GE* that has *GE* as a special case is not testable. This includes hypotheses of imperfect competition, transaction costs, borrowing constraints, incomplete markets, and deviations from rationality. (A similar point is made in Friedman's (1953) criticism of monopolistic competition.)

Not only are *cGE* simpler models than *GE* models, we will show that they are *better*. First the equilibrium of a *cGE* model is always unique. Second, computation is much easier with guaranteed success. No specialized algorithms are needed, such as those of those of Scarf (1973), nor does one need to calculate or approximate derivatives, as with quasi-Newton methods. Third, the price vector is the fixed point of a *linear* mapping and so questions of existence can be answered using linear algebra. Fourth, if one wishes one can dispense with real analysis. That is it is possible to construct general equilibrium theory entirely within the rational numbers $\mathbb{Q}$ without any reference to $\mathbb{R}$. This makes the *cGE* model *more* realistic than the *GE* model since all prices and quantities in the real economic world are in $\mathbb{Q}$.

2 *cGE* Models

Consider an economy with a set $\mathcal{H} = \{1, \ldots, H\}$ of H households indexed by i, a set $\mathcal{G} = \{1, \ldots, G\}$ of G goods indexed by j, and a set $\mathcal{F} = \{1, \ldots, F\}$ of F factors of production indexed by l. The total amount of good $j \in \mathcal{G}$ is Q_j with price P_j. Total expenditure is $Y \equiv \sum_{j=1}^{G} P_j Q_j$. Define $p_j \equiv \frac{P_j Q_j}{Y}$, the $1 \times G$ row vector $p \equiv [p_j]$, and the $n \times 1$ column vector of ones $\iota_n \equiv [1]$. We normalize prices so that $Y = 1$ so that p is in the unit simplex as

$$p\iota_G = 1.$$

The total amount of factor $l \in \mathcal{F}$ used in producing good j is K_{jl}. The price of factor l is W_l with total supply $K_l \equiv \sum_{j=1}^{G} K_{jl}$, which we normalize so that $K_l = 1$. Define $\omega_l \equiv \frac{W_l K_l}{Y}$ and the $1 \times F$ vector $\omega \equiv [\omega_l]$ where ω_l is factor $l's$ share of total income Y.

Goods are produced with a constant returns to scale Cobb-Douglas production function

$$Q_j = \Gamma_j \prod_{l=1}^{F} K_{jl}^{\beta_{jl}} \text{ with } \beta_{jl} \geq 0 \text{ and } \sum_{l=1}^{F} \beta_{jl} = 1. \tag{1}$$

Define the $G \times F$ matrix $B \equiv \left[\beta_{jl}\right]$. Constant returns to scale means that B is a stochastic matrix, that is it has non-negative elements with rows that sum to one, with[1]

$$B\iota_F = \iota_G. \tag{3}$$

The scale parameter Γ_j in (1) plays no essential role and so can be chosen for convenience as

$$\Gamma_j \equiv \frac{\hat{Q}_j}{p_j} \prod_{l=1}^{F} \left(\frac{\omega_l}{\beta_{jl}}\right)^{\beta_{jl}} \tag{4}$$

so that

$$Q_j = \hat{Q}_j \prod_{l=1}^{F} \left(\frac{\omega_l K_{jl}}{p_j \beta_{jl}}\right)^{\beta_{jl}}. \tag{5}$$

The first-order conditions for profit maximization require that

$$K_{jl} = \frac{p_j \beta_{jl}}{\omega_l} \tag{6}$$

[1] Intermediate goods, that is goods sold by firms to other firms, can be accounted for by expressing B as a product of stochastic matrices as

$$B = B(n) B(n-1) \cdots B(1). \tag{2}$$

Given (2) there are n layers of firms, with firms of layer j producing goods for firms of layer $j+1$, where $B(j)$ plays the role of B in (1) in the production function of each layer, and where layer 1 firms use $\bar{K}_l$ for $l \in \mathcal{F}$ as inputs, and layer n firms produce final goods Q_j for $j \in \mathcal{G}$. Our arguments do not require the factorization in (2), and so we leave the details to the Appendix.

so $Q_j = \hat{Q}_j$ in equilibrium. For the theoretical model we make the normalization $\hat{Q}_j = 1$ in (4) so that $Q_j = 1$ in equilibrium and

$$p_j \equiv \frac{P_j Q_j}{Y} = P_j Q_j = P_j$$

and so we can identify p_j as either industry $j's$ share of total expenditure Y, the revenue of industry j, or the price of good j.

Summing over j in (6) and using $K_l = 1$ we have

$$\omega = pB. \tag{7}$$

It follows that ω is a convex combination of the rows of B, and is on the unit simplex since

$$\omega\iota_F = pB\iota_F = p\iota_G = 1.$$

Household preferences are assumed to be Cobb-Douglas, expressed in logarithmic form, with utility for household i given by

$$U_i = \sum_{j=1}^{G} \alpha_{ij} \ln(Q_{ij}) \text{ with } \alpha_{ij} \geq 0 \text{ and} \sum_{j=1}^{G} \alpha_{ij} = 1 \tag{8}$$

where Q_{ij} is household $i's$ consumption of good j. Define the $H \times G$ matrix $A \equiv [\alpha_{ij}]$, which is a stochastic matrix with

$$A\iota_G = \iota_H. \tag{9}$$

Household i has an endowment E_{li} of factor l with endowment share $\varepsilon_{li} \equiv \frac{E_{li}}{E_l}$ of $E_l \equiv \sum_{i=1}^{H} E_{li}$. Since $E_l = K_l = 1$, we have $\varepsilon_{li} = E_{li}$ and the $F \times H$ matrix $E \equiv [\varepsilon_{li}]$ is a stochastic matrix or

$$E\iota_H = \iota_F. \tag{10}$$

The income of household i is $Y_i \equiv \sum_{l=1}^{F} W_l E_{li}$ so that if $y_i \equiv \frac{Y_i}{Y}$ is household $i's$ income share, then $y_i = \sum_{l=1}^{F} \omega_l \varepsilon_{li}$. Defining the $1 \times H$ vector $y \equiv [y_i]$ we have

$$y = \omega E \tag{11}$$

and so y is a convex combination of the rows of E, and is on the unit simplex since

$$y\iota_H = \omega E \iota_H = \omega\iota_F = 1.$$

The first-order conditions for utility maximization require that $P_j Q_{ij} = Y_i \alpha_{ij}$. Define $q_{ij} \equiv \frac{Q_{ij}}{Q_j}$ as the share of good j going to household i. Using the normalizations $Q_j = Y = 1$ we have

$$q_{ij} = Q_{ij} = \frac{y_i \alpha_{ij}}{p_j} \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}. \tag{12}$$

So we can also interpret Q_{ij} as the share of good j that household i consumes. Summing (12) over i and using $\sum_{i=1}^{H} Q_{ij} = Q_j = 1$ we have

$$p = yA. \tag{13}$$

It follows that p is a convex combination of the rows of A.

The cGE model reduces to an exchange economy if $F = G$ and $B = I$, in which case $\omega = pB$ reduces to $\omega = p$.

2.1 Generality of the *cGE* Model

In specifying the cGE model we have not mentioned time t, location l, or state of the world s. But this does not restrict the generality of the cGE model because the number of households H, goods G , and factors F is unrestricted. So we are free to index all these by time, location, and state of the world, as in Debreu (1979). So for example if we take the G goods indexed by the integer j as $j = g, t, l, s$ where g indexes a type of good, t indexes a time period, l indexes a location, and s indexes a state of the world, then Q_j would denote production of good type g at time period t in location l in state of the world s. To be more specific suppose that the integer j is

$$j = 507, 912, 398, 865.$$

The first three digits $g = 507$ might correspond to a type of wheat, the next three digits $t = 912$ might correspond to the time of August of the year 1955, the next three digits $l = 398$ might correspond to the location North Dakota, and the final three digits $s = 865$ might correspond to a particular weather pattern. So Q_j would be the production of a particular type of wheat in August of 1955 in North Dakota given a particular weather pattern. If G is large enough we can have production of goods for as precise a description of time, location, and state of the world as we like.

2.2 Regularity Assumptions

We now show that the following three regularity assumptions insure a unique equilibrium with positive prices and incomes. In Section 2.5 and the Appendix we show that the resulting allocation is still unique even if these assumptions do not hold and the equilibrium price vector is not unique.

Assumption 1: For all $i \in \mathcal{H}$ there is an $l \in \mathcal{F}$ such that $\varepsilon_{li} > 0$.
Assumption 2: For all $l \in \mathcal{F}$ there is a $j \in \mathcal{G}$ such that $\beta_{jl} > 0$.
Assumption 3: The $G \times G$ matrix $C \equiv BEA$ is indecomposable.

Assumptions 1 and 2 state that each household owns a positive endowment of some factor and each factor of production is required in the production of some good.

Assumption 3 imposes a minimal level of interdependence in the economy, in the sense explained below, that all goods are linked to each other either directly or indirectly through consumption and production. We say good j_1 is directly linked to good j_2, written as $j_1 \Longrightarrow j_2$, if there is some household i that both demands good j_2 and supplies a factor of production l used in the production of good j_1, that is if there is an $i \in \mathcal{H}, l \in \mathcal{F}$ such that

$$\beta_{j_1 l} \varepsilon_{li} \alpha_{ij_2} > 0.$$

We then say that good j_1 is linked (either directly or indirectly) to good j_n, written as $j_1 \rightarrow j_n$, if there exists a sequence $j_2, \ldots, j_{n-1} \in \mathcal{G}$ such that

$$j_1 \Longrightarrow j_2 \Longrightarrow \cdots \Longrightarrow j_n.$$

Assumption 3 is then equivalent to assuming that all goods are linked to each other or

$$j_1 \rightarrow j_2 \text{ for all } j_1, j_2 \in \mathcal{G}.$$

2.3 Equilibrium

From (7), (11), (13) the equilibrium price vector p satisfies

$$p = pC \text{ and } p\iota_G = 1 \text{ where } C \equiv BEA. \tag{14}$$

So p is the fixed point of a mapping $p = pC$ from the unit simplex onto itself. Unlike other GE models, this mapping is linear.

The matrix $C \equiv BEA$ that defines this mapping is composed, via matrix multiplication, of the three essential components of a general equilibrium economy: the technology B, endowments E, and tastes A. By a permutation of the order of multiplication it is possible to obtain a mapping that determines wages $\omega = pB$ via $\omega = \omega C_\omega$ where $C_\omega \equiv EAB$, or incomes $y = \omega E$ via $y = yC_y$ where $C_y \equiv ABE$.

Given strictly positive p, ω, y the resulting allocation of resources is $\{Q, K\}$ as described by the $H \times G$ matrix $Q = [Q_{ij}]$ and the $G \times F$ matrix $K = [K_{lj}]$ where

$$Q_{ij} = \frac{y_i \alpha_{ij}}{p_j}, K_{jl} = \frac{p_j \beta_{jl}}{\omega_l} \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}. \tag{15}$$

2.4 Social Welfare

The allocation in (15) can also be generated by the optimization of an additive social welfare function with weights given by income vector y. From (5) we have

$$\ln(Q_j) = \ln(\Gamma_j) + \sum_{l=1}^{F} \beta_{jl} \ln(K_{jl})$$

so that ignoring constants social welfare U is

$$U=\sum_{i=1}^{H}y_iU_i=\sum_{i=1}^{H}y_i\sum_{j=1}^{G}\alpha_{ij}\ln\left(Q_{ij}\right)+\sum_{j=1}^{G}p_j\sum_{l=1}^{F}\beta_{jl}\ln\left(K_{jl}\right).$$

Maximizing U subject to the constraints $\sum_{i=1}^{H}Q_{ij}=1$ and $\sum_{j=1}^{G}K_{jl}=1$ then yields (15).

2.5 Regularity Assumptions are not Essential

If Assumption 3 is violated and the matrix C is decomposable, then the solution to $p=pC$ is not necessarily unique. Unlike *GE* models however, it turns out that this non-uniqueness is of no economic importance: it simply reflects the fact that the economy can be broken down into a set of smaller non-interacting island economies, with different solutions corresponding to arbitrary nominal normalizations of prices of non-traded goods between these islands. This is reflected in the fact, shown in the Appendix, that even though p may not be unique, the resulting allocation of resources $\{Q,K\}$ is unique.

The matrix C is decomposable if there is a partition $\mathcal{G}=\mathcal{G}_1\cup\mathcal{G}_2$ with $G_1>0$ goods in $\mathcal{G}_1$ and $G_2>0$ goods in $\mathcal{G}_2$, such that if we list the goods in $\mathcal{G}_1$ first and the goods in $\mathcal{G}_2$ second, then C can be written as

$$C=\begin{bmatrix}C_{11} & 0\\ C_{21} & C_{22}\end{bmatrix}. \tag{16}$$

We now show the two types of outcomes that can occur when Assumption 3 is violated.

The first type of outcome occurs when $C_{21}\neq 0$ with C_{11} indecomposable in (16). Then $p=pC$ has a unique solution

$$p=\begin{bmatrix}p^1 & 0\end{bmatrix}$$

where $p^1C_{11}=p^1$ with $p^1\iota_{G_1}=1$. All that happens in this case is that the goods in $\mathcal{G}_2$ have a zero price.

The second type of outcome occurs when $C_{21}=0$ with C_{11} and C_{22} both indecomposable as

$$C=\begin{bmatrix}C_{11} & 0\\ 0 & C_{22}\end{bmatrix}. \tag{17}$$

This leads to multiple solutions for p. The reason for this is that the economy can be split into two non-interacting island economies with goods $\mathcal{G}_1$ in island economy one, and goods $\mathcal{G}_2$ in island economy 2. The competitive solution to each individual island economy is strictly positive and unique and takes the form $\tilde{p}^1C_{11}=\tilde{p}^1$ and $\tilde{p}^2C_{22}=\tilde{p}^2$ with $\tilde{p}^1\iota_{G_1}=1$ and $\tilde{p}^2\iota_{G_2}=1$. The competitive solution p for the entire economy is however not unique, and takes the form

$$p=\begin{bmatrix}\delta\tilde{p}^1 & (1-\delta)\,\tilde{p}^2\end{bmatrix}\text{ for any }0<\delta<1.$$

Different solutions correspond to different values of δ. But δ only determines the relative prices between the two islands, and since these islands do not trade with each other, the choice of δ does not affect the resulting allocation.

A more detailed analysis is carried out in the Appendix. But these two cases capture the essential possibilities: either there are zero prices or incomes, or there is non-uniqueness due to the arbitrarily normalization of non-traded goods between non-interacting island economies. Neither has any consequences for the uniqueness of the resulting allocation of resources.

3 Fitting the *cGE* Model to Data

In this section we will show that the *cGE* model can rationalize any conceivable data set of prices and quantities. The essence of the argument is as follows: Suppose for all locations, time periods, and states of the world we are given the prices of all goods and factors of production, the endowments, incomes, and consumption of all goods of all households, and the factor usage and production levels of all industries. Following Debreu (1979), we index goods and factors to take into account all possible locations, times, and states of the world. We then match factor and budget shares in the model with the factor and budget shares observed in the data. The model will then predict exactly the observed set of data on the assumption that the data is internally consistent.

Where it is convenient we use a $^+$ to distinguish variables in the data set from those in the model. For example Q_j^+ is total production of good j in the data while Q_j is value of total production predicted by the model.

Data Set: $P_j^+, W_l^+, E_{li}^+, Q_{ij}^+, Q_j^+, K_{jl}^+$ for all $i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}$.

From this data set we define for all $i \in \mathcal{H}, l \in \mathcal{F}$

$$K_l^+ \equiv \sum_{j=1}^{G} K_{jl}^+, E_l^+ \equiv \sum_{i=1}^{H} E_{li}^+, Y_i^+ \equiv \sum_{j=1}^{G} P_j^+ Q_{ij}^+,\ Y^+ \equiv \sum_{i=1}^{H} Y_i^+.$$

In addition we define the sample analogues of p, ω, y: the $1 \times G$ vector $p^+ \equiv \left[p_j^+\right]$, the $1 \times F$ vector $\omega^+ \equiv \left[\omega_l^+\right]$, and the $1 \times H$ vector $y^+ \equiv \left[y_i^+\right]$ as

$$p_j^+ \equiv \frac{P_j^+ Q_j^+}{Y^+}, \omega_l^+ \equiv \frac{W_l^+ K_l^+}{Y^+}, y_i^+ \equiv \frac{Y_i}{Y^+} \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}. \tag{18}$$

We assume all data prices are strictly greater than 0, and that the data satisfy four consistency conditions.

Data Assumption 1: $Y_i^+ = \sum_{l=1}^{F} W_l^+ E_{li}^+$ *for all* $i \in \mathcal{H}$.
Data Assumption 2: $E_l^+ = K_l^+$ *for all* $l \in \mathcal{F}$.
Data Assumption 3: $Q_j^+ = \sum_{i=1}^{H} Q_{ij}^+$ *for all* $j \in \mathcal{G}$.
Data Assumption 4: $P_j^+ Q_j^+ = \sum_{l=1}^{F} W_l^+ K_{jl}^+$ *for all* $j \in \mathcal{G}$.

The first Data Assumption states that households respect their budget constraints; the second that total household endowments equal the total factor use; the third that total household consumption equals industry production; the fourth that each industry makes zero profits. Essentially these are accounting identities, unrelated to the Cobb-Douglas assumption. Real world data sets might violate these conditions, but they could always be modified so they do not.[2]

Using this data set we match a cGE model with $A = [\alpha_{ij}]$, $B = [\beta_{jl}]$, $E = [\varepsilon_{li}]$ as follows

$$\alpha_{ij} \equiv \frac{P_j^+ Q_{ij}^+}{Y_i^+}, \beta_{jl} \equiv \frac{W_l^+ K_{jl}^+}{P_j^+ Q_j^+}, \varepsilon_{li} \equiv \frac{E_{li}^+}{E_l^+} \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}.$$

We set $\hat{Q}_j = Q_j^+$ in (4) as

$$\Gamma_j \equiv \frac{Q_j^+}{p_j^+} \prod_{l=1}^{F} \left(\frac{\omega_l^+}{\beta_{jl}} \right)^{\beta_{jl}}$$

from which it will follow that $Q_j = Q_j^+$, and so the model correctly predicts industry production. If C were decomposable order r we could, following the argument in the Appendix, break up the fitted model into r non-interacting islands where each C matrix would then be indecomposable. We assume then, without loss of generality, that the matrix $C = ABE$ is indecomposable.

We first show that the estimated matrices A, B, E are stochastic matrices, and so are consistent with the assumptions of the model. Clearly the elements of these matrices are all non-negative so what needs to be shown is that all rows sum to one. To this end we have $A\iota_G = \iota_H$ since

$$\sum_{j=1}^{G} \alpha_{ij} = \frac{\sum_{j=1}^{G} P_j^+ Q_{ij}^+}{Y_i^+} \equiv 1$$

by the definition of Y_i^+. We have $E\iota_H = \iota_F$ by the definition of E_l^+ since

$$\sum_{i=1}^{H} \varepsilon_{li} = \frac{\sum_{i=1}^{H} E_{li}^+}{E_l^+} \equiv 1.$$

To show that $B\iota_F = \iota_G$ we use Data Assumption 4 as

$$\sum_{l=1}^{F} \beta_{jl} = \frac{\sum_{l=1}^{F} W_l^+ K_{jl}^+}{P_j^+ Q_j^+} = 1.$$

[2] For example if some industry had positive profits so that Data Assumption 4 appeared to be violated, it would always be possible to assume an unobserved factor, owned by shareholders, to which the residual profits are paid.

We now show that the analogues of p, y, ω given by p^+, y^+, ω^+ in (18) satisfy the predicted relations from the model

$$\begin{aligned} y^+ A &= p^+, p^+ B = \omega^+, \omega^+ E = y^+ \\ p^+ \iota_G &= 1, y^+ \iota_H = 1, \omega^+ \iota_F = 1. \end{aligned}$$

Clearly the elements of p^+, y^+, ω^+ are non-negative as required by the model. To show $\omega^+ E = y^+$ we have

$$\sum_{l=1}^{F} \omega_l^+ \varepsilon_{li} = \sum_{l=1}^{F} \frac{W_l^+ K_l^+}{Y^+} \frac{E_{li}^+}{E_l^+} = \frac{\sum_{l=1}^{F} W_l^+ E_{li}^+}{Y^+} = \frac{Y_i^+}{Y^+} \equiv y_i^+$$

where we have used Data Assumptions 1 and 2. To show $p^+ B = \omega^+$ we have

$$\sum_{j=1}^{G} p_j^+ \beta_{jl} = \sum_{j=1}^{G} \frac{P_j^+ Q_j^+}{Y^+} \frac{W_l^+ K_{jl}^+}{P_j^+ Q_j^+} = \frac{W_l^+ \sum_{j=1}^{G} K_{jl}^+}{Y^+} = \frac{W_l^+ K_l^+}{Y^+} \equiv \omega_l^+$$

where we have used the definition of K_l^+. To show $y^+ A = p^+$ we have

$$\sum_{i=1}^{H} y_i^+ \alpha_{ij} = \sum_{i=1}^{H} \frac{Y_i^+}{Y^+} \frac{P_j^+ Q_{ij}^+}{Y_i^+} = \frac{P_j^+ \sum_{i=1}^{H} Q_{ij}^+}{Y^+} = \frac{P_j^+ Q_j^+}{Y^+} \equiv p_j^+$$

where we have used Data Assumption 3. We have $y^+ \iota_H = 1$ by the definition of Y^+. Since $y^+ A = p^+$ and $A\iota_H = \iota_G$, it follows that $p^+ \iota_G = 1$. Since $p^+ B = \omega^+$ and $B\iota_G = \iota_F$, it follows that $\omega^+ \iota_F = 1$.

So $p = p^+$ is a solution to $pC = p$ and $p\iota_G = 1$ and hence the fitted model predicts the observed incomes and prices $p = p^+, y = y^+, \omega = \omega^+$. Since C is indecomposable, it follows that $p = p^+, y = y^+, \omega = \omega^+$ is the unique solution. From the solution $p = p^+, y = y^+, \omega = \omega^+$ comes a unique allocation of resources, which is precisely the observed allocation of resources, as we now show. The model correctly predicts the proportion of factor l going to industry j as

$$K_{jl} = \frac{p_j \beta_{jl}}{\omega_l} = \frac{\frac{P_j^+ Q_j^+}{Y^+} \frac{W_l^+ K_{jl}^+}{P_j^+ Q_j^+}}{\frac{W_l^+ E_l^+}{Y^+}} = \frac{K_{jl}^+}{E_l^+} = \frac{K_{jl}^+}{K_l^+}$$

where we have used Data Assumption 2. The model correctly predicts the share of good j that goes to household i as

$$\frac{Q_{ij}}{Q_j} = \frac{y_i \alpha_{ij}}{p_j} = \frac{\frac{Y_i^+}{Y^+} \frac{P_j^+ Q_{ij}^+}{Y_i^+}}{\frac{P_j^+ Q_j^+}{Y^+}} = \frac{Q_{ij}^+}{Q_j^+}.$$

Since we set $\hat{Q}_j = Q_j^+$ in (4) it follows that $Q_j = Q_j^+$ and so the model correctly predicts industry production. By setting the model's nominal income equal to the data's as $Y = Y^+$, it then follows that nominal prices in $p_j \equiv \frac{P_j Q_j}{Y}$ for all $j \in \mathcal{G}$ and $\omega_l \equiv \frac{W_l E_l}{Y}$ for all $l \in \mathcal{F}$ are correctly predicted.

In this derivation we have assumed that the data set contains output and factors at the industry level. It is conceivable that one would have data at the firm level, although what one means by a firm is ambiguous with constant returns to scale. If though the data set were more detailed with output and factors at the firm level, then this can also be incorporated as long as one is willing to have different Cobb-Douglas production functions for each firm, as shown in the Appendix.

3.1 Brown and Matzkin's Test of *GE*

Brown and Matzkin (1996) argue that the *GE* hypothesis is testable; that there are potential data sets that could falsify the *GE* hypothesis. This would appear to contradict our result that the *cGE* hypothesis, and hence the *GE* hypothesis, is not testable. The explanation for this apparent contradiction is that Brown and Matzkin assume a restricted version of the *GE* model, and it is these restrictions that can be tested, and not the *GE* hypothesis itself.

Brown and Matzkin assume data consisting of $N > 1$ observations from a *GE* exchange economy of price vectors p^t and endowments E^t for $t = 1, 2, \ldots, N$. Following Brown and Matzkin we will think of these $N > 1$ observations as N cities, although they could equally well be different time periods or states of the world.

The problem is that an unrestricted *GE* model would only generate one observation (i.e., $N = 1$). This is because in an unrestricted *GE* model we can index households and goods by city, and so the unrestricted *GE* model would generate exactly $N = 1$ observations of p and E. Restrictions must be imposed in order for $N > 1$ observations in the Brown and Matzkin setup to be possible. Households could only have endowments of goods from their own cities, and they could only demand goods from their own cities. As well their preferences would have to be identical across cities.

The nature of the implicit restrictions Brown and Matzkin impose can be illustrated using a *cGE* exchange economy with $F = G$ and $B = I$. We partition goods $\mathcal{G}$ and households $\mathcal{H}$ by the $N > 1$ cities. In order for households to only have endowments from their city, to only consume goods from their city, and to have the same preferences in each city, E and A need to be restricted to being block diagonal as

$$E = \begin{bmatrix} E^1 & 0 & \cdots & 0 \\ 0 & E^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & E^N \end{bmatrix} \text{ and } A = \begin{bmatrix} \tilde{A} & 0 & \cdots & 0 \\ 0 & \tilde{A} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A} \end{bmatrix}$$

so that C takes the block diagonal form

$$C = BEA = EA = \begin{bmatrix} E^1 & 0 & \cdots & 0 \\ 0 & E^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & E^N \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 & \cdots & 0 \\ 0 & \tilde{A} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A} \end{bmatrix} = \begin{bmatrix} E^1\tilde{A} & 0 & \cdots & 0 \\ 0 & E^2\tilde{A} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & E^N\tilde{A} \end{bmatrix}.$$

With these restrictions on E and A we obtain N price vectors satisfying $p^t = p^t E^t \tilde{A}$ along with the endowments E^t for $t = 1, \ldots, N$. Without these restrictions we only obtain one observation satisfying $p = pC$ for endowment E.

4 Computation

Computing solutions to GE models is generally difficult and requires specialized software that uses algorithms such as that in Scarf (1973), or quasi-Newton methods as discussed in Shoven and Whalley (1984). Convergence is guaranteed for Scarf's algorithm, but in practice is too slow. Quasi-Newton methods usually converge faster, but do not have guaranteed convergence, and require the calculation of derivatives.

Computing solutions to cGE models on the other hand is easy. Derivatives are not required. There is an iterative algorithm where geometric convergence is guaranteed. Alternatively an exact solution can be computed in a finite number of steps with or without matrix inversion.

The first method is to iterate as $p^t = p^{t-1}C$ with any non-negative initial p^0 satisfying $p^0 \iota_G = 1$. As long as C is acyclic, that is as long as it has no eigenvalues besides $\lambda = 1$ on the unit circle, then $p^t = p + O\left(\delta^t\right)$ where $\delta = |\lambda_2| < 1$ and where λ_2 is the second largest eigenvalue of C.[3] If C is cyclic of order s, then convergence can be obtained by replacing C with with C^s as $p^t = p^{t-1}C^s$.

The second method is to treat $p = pC$ and $p\iota_G = 1$ as a system of linear equations and find an exact solution using matrix inversion. Partition C and p as

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \text{ and } p = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$$

where C_{22} and p_2 are scalars. First assume $C_{12} \neq 0$. Then

$$(I - C_{11})^{-1} = \sum_{k=0}^{\infty} C_{11}^k$$

exists with non-negative elements. From

$$p_1 C_{11} + p_2 C_{21} = p_1 \text{ and } p_1 C_{12} + p_2 C_{22} = p_2$$

[3] If C is cyclic then (see Debreu and Herstein (1953)) the economy, or an island component, has a strictly seasonal structure with period s and eigenvalues $\lambda_k = e^{\frac{2\pi i}{s}k}$ for $k = 1, 2, \ldots, s-1$. In this case p^t will not converge to p but instead cycle with a period of s.

we have $p_1 = p_2 C_{21} (I - C_{11})^{-1}$ so that using $p_1 \iota_{G-1} + p_2 = 1$ yields

$$p_1 = \frac{C_{21} (I - C_{11})^{-1}}{1 + C_{21} (I - C_{11})^{-1} \iota_{G-1}}, p_2 = \frac{1}{1 + C_{21} (I - C_{11})^{-1} \iota_{G-1}}. \tag{19}$$

If $C_{12} = 0$ then C_{11} is a stochastic matrix with $p_1 = p_1 C_{11}$, and so p_1 can be calculated using the same approach as p. If $C_{22} \neq 1$ then $p_2 = 0$, while if $C_{22} = 1$ then there will be multiple solutions for p of the form

$$p = \left[\begin{array}{cc} (1-\delta) p_1 & \delta \end{array} \right]$$

for any $0 < \delta < 1$. Since δ has no affect on the allocation, we can simply pick anything convenient like $\delta = \frac{1}{2}$.

A third method for solving $p = pC$ that does not require matrix inversion is described in the Appendix.

With any of these methods the computational difficulty of solving $p = pC$ is determined by the number of goods G. If either the number of factors F or the number of households H is smaller, then it is possible to reduce the dimension of the problem by instead solving for ω or y first. If F is the smallest, then multiply both sides of $p = pC$ by B and use $\omega = pB$ to obtain $\omega = \omega C_\omega$ where $C_\omega \equiv EAB$ is an $F \times F$ stochastic matrix, and where ω can be solved for in the same manner as $p = pC$. If H is the smallest, multiply both sides of $\omega = \omega C_\omega$ by E and use $y = \omega E$ to obtain $y = y C_y$ where $C_y \equiv ABE$ is an $H \times H$ stochastic matrix.

4.1 Pareto Efficiency

Consider any feasible allocation of resources $\{Q, K\}$ where the $H \times G$ and $G \times F$ matrices of good and factor shares $Q \equiv [Q_{ij}]$ and $K \equiv [K_{jl}]$ satisfy $Q^T \iota_H = \iota_G$ and $K^T \iota_G = \iota_F$. From solving $p = pC$ we can easily solve the following two problems: First, if $\{Q, K\}$ is Pareto optimal, then find the competitive equilibrium that supports it. Second, if $\{Q, K\}$ is *not* Pareto optimal, then find a competitive equilibrium that Pareto dominates $\{Q, K\}$; that is where all households are at least as well off, and at least one household is strictly better off. In Sampson (2013, 2) the solution to this second problem is used to find an efficient tax regime that Pareto dominates an inefficient tax regime; that is where there are no losers when one moves to the efficient tax regime.

Both of these problems are solved by considering a competitive economy with endowment matrix $E = K^T Q^T$, which is a stochastic matrix since

$$E \iota_H = K^T Q^T \iota_H = K^T \iota_G = \iota_F.$$

Solving $p = pC$ for $p \iota_G = 1$ with $\omega = pB$ and $y = \omega E$, we have

$$y = \omega E = \omega K^T Q^T \geq p Q^T$$

where $\omega K^T \geq p$ follows from the fact that in a competitive equilibrium with constant returns to scale, each feasible production plan must yield either 0 or negative profits. It follows that in the competitive equilibrium each household can afford to consume its consumption levels in the initial allocation $\{Q, K\}$, and so must be at least as well off. If $\{Q, K\}$ is Pareto optimal, then since the profit and utility maximization solutions are unique, it follows that $Q_{ij} = \frac{y_i \alpha_{ij}}{p_j}, K_{jl} = \frac{p_j \beta_{jl}}{\omega_l}$ for all $i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}$ and so the first problem is solved. If $\{Q, K\}$ is *not* Pareto optimal, then since the competitive equilibrium is Pareto optimal, it follows that all households are at least as well off, and at least one is must be strictly better off, and so the second problem is solved.

5 Imposing Restrictions on *cGE* Models

Although it is possible to find a *cGE* model that provides a perfect fit for any data set, a more useful *cGE* model would likely be more parsimonious, and so one would seek to impose restrictions. An advantage of *cGE* models is that they are linear, so that in imposing restrictions one can use the tools of matrix algebra in a manner similar to econometrics.

One type of restriction is imposing zero restrictions on the elements of A, B, E. If enough of these zero restrictions are made then A, B, E will be block diagonal as

$$A = \begin{bmatrix} A^1 & 0 \\ 0 & A^2 \end{bmatrix}, B = \begin{bmatrix} B^1 & 0 \\ 0 & B^2 \end{bmatrix}, E = \begin{bmatrix} E^1 & 0 \\ 0 & E^2 \end{bmatrix}.$$

In this case $C = BEA$ is decomposable, and the economy can be broken down into two smaller non-interacting island economies. We have also seen that block diagonality restrictions are required if one wishes to collect more than observation of prices and quantities from a *cGE* model, as with the Brown and Matzkin framework.

Another type of restriction that allows one to break a larger economy down into a set of smaller economies, but in a multiplicative manner, is to use the Kronecker product as

$$A = A^1 \otimes A^2, B = B^1 \otimes B^2, E = E^1 \otimes E^2$$

where A^m, B^m, E^m are $H_m \times G_m$, $G_m \times F_m$, $F_m \times H_m$ stochastic matrices for $m = 1, 2$ and where $H = H_1 H_2, G = G_1 G_2$, $F = F_1 F_2$. In this case

$$C = BEA = \left(B^1 \otimes B^2\right)\left(E^1 \otimes E^2\right)\left(A^1 \otimes A^2\right) = C^1 \otimes C^2$$

where $C^m \equiv B^m E^m A^m$ for $m = 1, 2$ are stochastic matrices since $\iota_H = \iota_{H_1} \otimes \iota_{H_2}$ so that

$$C\iota_H = C^1 \otimes C^2 \left(\iota_{H_1} \otimes \iota_{H_2}\right) = C^1 \iota_{H_1} \otimes C^2 \iota_{H_2} = \iota_{H_1} \otimes \iota_{H_2} = \iota_H.$$

Given the equilibrium for the two smaller economies: $p^m = p^m C^m$ and $p^m \iota_{G_m} =$

1 with $\omega^m = p^m B^m, y^m = \omega^m E^m$ for $m = 1, 2$, the equilibrium for the original economy $p = pC$ and $p\iota_G = 1$ with $\omega = pB, y = \omega E$ is then

$$p = p^1 \otimes p^2, \omega = \omega^1 \otimes \omega^2, y = y^1 \otimes y^2.$$

The resulting allocation will be $Q^m = \left[\frac{y_i^m \alpha_{ij}^m}{p_j^m}\right]$ and $K^m = \left[\frac{p_j^m \beta_{jl}^m}{\omega_l^m}\right]$ for $m = 1, 2$ with $Q = Q^1 \otimes Q^2$ and $K = K^1 \otimes K^2$.

Yet another approach is to use reduced rank restrictions. These restrictions have a natural interpretation as aggregation assumptions. Suppose for example we wish to reduce the dimension of a *cGE* model by aggregating the H households into $H_1 < H$ representative households. This can be accomplished by imposing the reduced rank restriction

$$A = \Gamma \tilde{A} = \sum_{m=1}^{H_1} \gamma^m \tilde{\alpha}^m$$

where $\Gamma \equiv [\gamma^m]$ is an $H \times H_1$ stochastic matrix with columns γ^m satisfying $\sum_{m=1}^{H_1} \gamma^m = \iota_H$, and $\tilde{A} \equiv [\tilde{\alpha}^m]$ is an $H_1 \times G$ stochastic matrix with rows $\tilde{\alpha}^m$. Below we show that the matrix Γ aggregates the H households into the H_1 representative households with tastes described by $\tilde{A}$, incomes by $\tilde{y} = y\Gamma$, and endowments by $\tilde{E} \equiv E\Gamma$. The fact that $A = \sum_{m=1}^{H_1} \gamma^m \tilde{\alpha}^m$ means that each of the H households in the original large economy has preferences that are convex combinations of the preferences of the H_1 representative households. The $1 \times H_1$ vector $\tilde{H}$ given by

$$\tilde{H} \equiv \iota_H^T \Gamma = \left[\iota_H^T \gamma^m\right] = \left[\tilde{H}^m\right]$$

where $\tilde{H}^m$ is the population of the type m representative households with $\sum_{m=1}^{H_1} \tilde{H}^m = H$.

Equilibrium income y for the original economy satisfies $y = yC_y$ where $C_y \equiv ABE$ is an $H \times H$ stochastic matrix. Using $A = \Gamma\tilde{A}$ and $\tilde{E} \equiv E\Gamma$ we have, by multiplying both sides of $y = y\Gamma\tilde{A}BE$ by Γ that $\tilde{y} = \tilde{y}C_{\tilde{y}}$ where $C_{\tilde{y}} \equiv \tilde{A}B\tilde{E}$ is an $H_1 \times H_1$ stochastic matrix. Given the equilibrium vector of incomes $\tilde{y}$ for the H_1 representative households, we obtain the vector of equilibrium prices as $p = \tilde{y}\tilde{A}$ and wages as $\omega = pB$. To recover the incomes for the original H households we then use $y = \omega E$.

For $H_1 = 1$ we have $\Gamma = \iota_H$ and $\tilde{A} = \tilde{\alpha}$ so that there is a single representative household with $A = \iota_H \tilde{\alpha}$, where $\tilde{\alpha}$ gives the utility weights for the representative household.[4] Solving yields $\tilde{y} = 1$, $p = \tilde{\alpha}$, $\omega = \tilde{\alpha}B, y = \omega E$. Since $p_j = \alpha_{ij} = \tilde{\alpha}_j$, consumption is identical to income or $Q_{ij} = \frac{y_i \alpha_{ij}}{p_j} = y_i$.

Using exactly the same approach, reduced rank restrictions can also be used to aggregate factors and goods.

[4] This representative agent model could be falsified by the idealized data set (just find two or more households that have different budget shares for at least one good.) But if the data did not contain individual household consumption (the Q_{ij}), then the representative household model could not be falsified.

These strategies can be combined when imposing restrictions. For example with the Kronecker product representation we can also assume a representative household for the second economy so that $A_2 = \iota_{H_2}\alpha^2$ where α^2 is $1 \times G_2$. Then

$$C = BEA = \left(B^1 \otimes B^2\right)\left(E^1 \otimes E^2\right)\left(A^1 \otimes \iota_{H_2}\alpha^2\right) = C^1 \otimes \iota_{G_2}\alpha^2$$

from which it follows that $p = p^1 \otimes \alpha^2$.

6 Rational Economics

> *Why should I believe in a real number if I can't calculate it, if I can't prove what its bits are, and if I can't even refer to it? And each of these things happens with probability one!* -Chaitin (2006).

Economics in $\mathbb{R}$ is unrealistic: Infinite precision is impossible in either prices or quantities. Only a countable subset of the uncountable elements of $\mathbb{R}$ can even be computed. For *GE* models $\mathbb{R}$ is in general unavoidable. But a theory of general equilibrium can be constructed within $\mathbb{Q}$ using *cGE* models, and thus avoid the unrealistic features of $\mathbb{R}$.

The idealized data, like any real world data set, has all its observations in $\mathbb{Q}$. It follows that all the parameters of the fitted *cGE* model are also in $\mathbb{Q}$; that is $A \in \mathbb{Q}^{H\times G}$, $B \in \mathbb{Q}^{G\times F}$, $E \in \mathbb{Q}^{F\times H}$, and so $C = BEA \in \mathbb{Q}^{G\times G}$. Since the vector of prices p is a fixed point of the linear mapping $p = pC$ with a solution p given by (19), it follows that $p \in \mathbb{Q}^G$. Since $p \in \mathbb{Q}^G$ it follows that $\omega = pB \in \mathbb{Q}^F$ and $y = \omega E \in \mathbb{Q}^H$. Since $p \in \mathbb{Q}^G, \omega \in \mathbb{Q}^F, y \in \mathbb{Q}^H$ it follows that $Q_{ij} = \frac{y_i\alpha_{ij}}{p_j} \in \mathbb{Q}$ and $K_{jl} = \frac{p_j\beta_{jl}}{\omega_l} \in \mathbb{Q}$ for all $i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}$. If $\hat{Q}_j \in \mathbb{Q}$ for all $j \in \mathcal{G}$, then from (5)

$$Q_j = \hat{Q}_j \prod_{l=1}^{F}\left(\frac{\omega_l K_{jl}}{p_j\beta_{jl}}\right)^{\beta_{jl}} = \hat{Q}_j \in \mathbb{Q}.$$

As long as the parameters of a *cGE* model are all in $\mathbb{Q}$, then all equilibrium prices and quantities will also be in $\mathbb{Q}$. So a theory of general equilibrium is possible within the rationals $\mathbb{Q}$.

While all observables are in $\mathbb{Q}$ for *cGE* models, there are places where it is convenient to bring in $\mathbb{R}$ as a means of simplifying derivations. Consider first the production function given above. Even though $Q_j \in \mathbb{Q}$ for the $K_j \equiv [K_{jl}] \in \mathbb{Q}^F$ that firms ultimately select, in general $Q_j \notin \mathbb{Q}$ for any arbitrary $K_j \in \mathbb{Q}^F$ that firms *might* select. A pragmatic approach would be to allow $Q_j \in \mathbb{R}$ and $K_j \in \mathbb{R}^F$, not because it is literally true, but because it makes the derivations simpler: the tools of calculus are at our disposal. Afterwards we can then verify that $K_j \in \mathbb{Q}^F$ and $Q_j \in \mathbb{Q}$.

It is possible to avoid any role for $\mathbb{R}$, but it would be more difficult and awkward, and might look something like this. Given $\beta_{jl} \in \mathbb{Q}$ it follows that $\beta_{jl} = \frac{\tilde{\beta}_{jl}}{n_j}$ where $\tilde{\beta}_{jl} \in \mathbb{N}$ and $n_j \in \mathbb{N}$ are integers. The production function relation above can then be rewritten as an implicit function $\phi_j(Q_j, K_j) = 0$ where

$$\phi_j(Q_j, K_j) = Q_j^{n_j} - \hat{Q}_j^{n_j} \prod_{l=1}^{F} \left(\frac{\omega_l K_{jl}}{p_j \beta_{jl}} \right)^{\tilde{\beta}_{jl}}.$$

Here $\phi_j(Q_j, K_j) \in \mathbb{Q}$ as long as $Q_j \in \mathbb{Q}, K_j \in \mathbb{Q}^F$, and so we are not required to use $\mathbb{R}$ when calculating $\phi_j(Q_j, K_j)$. There is still the problem that there will not in general exist a $Q_j \in \mathbb{Q}$ that results in $\phi_j(Q_j, K_j) = 0$ for any $K_j \in \mathbb{Q}^F$. But this can be remedied if we instead assume that firms in industry j optimize over the $Q_j \in \mathbb{Q}, K_j \in \mathbb{Q}^F$ that satisfy $\phi_j(Q_j, K_j) \leq 0$.

Another similar problem occurs for welfare since $U_i \notin \mathbb{Q}$ even with $Q_{ij} \in \mathbb{Q}$ since $\ln(Q_{ij}) \notin \mathbb{Q}$ in

$$U_i = \sum_{j=1}^{G} \alpha_{ij} \ln(Q_{ij}).$$

Again a pragmatic approach would be to allow $Q_{ij} \in \mathbb{R}$ and $U_i \in \mathbb{R}$ in the optimization, and then verify that when households optimize we have $Q_{ij} \in \mathbb{Q}$.

Alternatively one can avoid any mention of $\mathbb{R}$ by taking a monotonic transformation $\tilde{U}_i$ that insures that $\tilde{U}_i \in \mathbb{Q}$ whenever $Q_{ij} \in \mathbb{Q}$. Given $\alpha_{ij} \in \mathbb{Q}$ it follows that $\alpha_{ij} = \frac{\tilde{\alpha}_{ij}}{n_i}$ where $\tilde{\alpha}_{ij} \in \mathbb{N}$ and $n_i \in \mathbb{N}$ are integers. Then if $Q_{ij} \in \mathbb{Q}$ for all $j \in \mathcal{G}$ we have for all $i \in \mathcal{H}$

$$\tilde{U}_i \equiv e^{n_i U_i} = \prod_{j=1}^{G} Q_{ij}^{\tilde{\alpha}_{ij}} \in \mathbb{Q}.$$

7 Macroeconomics and Time Series Analysis

In macroeconomics *GE* and econometrics regularly meet, where dynamic *GE* (*DGE*) models are used. What would happen if macroeconomics were based on *cGE* models? This experiment has been tried: what one finds is that log prices and quantities follow vector autoregressions, as in Long and Plosser (1983) or Hercowitz and Sampson (1991). That is the *cGE* approach fits perfectly into modern time series econometrics. With *cGE* models one can use general equilibrium to link assumptions in the *cGE* model with stationarity, trends, unit roots and cointegration, as in Sampson (2013, 1). Scaling up the model with more households, factors, or goods is easy. The predictions arrived at will be as good as the predictions of the corresponding vector autoregression, which all evidence to date suggests is as good as can be hoped for.

The alternative experiment of using *DGE* models not based on *cGE* models began with Kydland and Prescott (1982), and continues to the present. Here we see the costs of complexity. No closed-form solution exists, and so numerical techniques have had to be used to approximately solve these models. The validity of these approximations is poorly understood. The predicted time series are not log-linear vector autoregressions, and so the relationship between economic theory and such basic properties as stationarity, trends, unit roots, and cointegration is poorly understood. The difficulties of numerical methods rise quickly as the dimension of the model increases, and so one must be content with a small number of households, factors, and goods. As one generalizes one will eventually have to confront the issue of the non-uniqueness of the equilibrium. The predictive power of these models has not been impressive, with any success coming from mimicking log-linear vector autoregressions, which are often used to approximate solutions. Because these models are poorly understood, economists who use them often don't know what they don't know, or have an exaggerated sense of their scientific success. Even though the claim is made that these are structural models, they do not appear to make truly structural predictions, such as being able to predict the recent 2008 financial crisis.

8 Conclusions

> *The supreme misfortune is when theory outstrips performance.* - Leonardo da Vinci

Cobb-Douglas in economics is special, just as inverse square is special in physics. In a wide range of different settings, Cobb-Douglas is the most natural and convenient place to begin. What we have shown in this paper is that nothing is gained by assuming anything more complicated than Cobb-Douglas, and much is lost. Equilibria are no longer unique, computation is much harder, and the model will make the unrealistic prediction that prices and quantities are in $\mathbb{R}\backslash\mathbb{Q}$. These additional difficulties come without any scientific benefit.

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10 Appendix

10.1 Intermediate Goods

Suppose we have firms selling their output to other firms. Suppose there are n layers of intermediate goods. Competitive firms in layer m buy $m-1$ level intermediate goods and use these to produce layer m intermediate goods. Let F_m be the number of intermediate goods at layer m. Layer 0 intermediate goods are the basic capital goods E_l for $l = 1, 2, \ldots, F_0$ where $F \equiv F_0$, and layer n goods are the final goods Q_j for $j = 1, 2, \ldots, F_n$ where $G \equiv F_n$. Production takes place as

$$\begin{aligned} K_j(m) &= \Gamma_j(m) \prod_{l=1}^{F_{m-1}} K_{jl}(m)^{\beta_{jl}(m)}, \beta_{jl}(m) \geq 0, \sum_{l=1}^{F_m} \beta_{jl}(m) = 1 \\ \sum_{j=1}^{F_m} K_{jl}(m) &= K_l(m-1) = 1, K_l(0) \equiv E_l, Q_j \equiv K_j(n) \end{aligned}$$

where $K_{jl}(m)$ is the amount of layer $m-1$ intermediate good l used to produce $K_j(m)$. The possibility that some level m^* intermediate good j is itself a final good can be accommodated by setting

$$\Gamma_j (m+1) = 1, \beta_{jj} (m+1) = 1 \text{ for } m \geq m^*$$

so that

$$K_j (m+1) = K_j (m) \text{ for } m \geq m^*.$$

Let $\omega_l (m)$ be the price of intermediate good l at layer m with $\omega_l (0) \equiv \omega_l$ and $\omega_j (n) \equiv p_j$. Profit maximization and $K_j (m) = 1$ implies that

$$\omega_j (m) \beta_{jl} (m) = \omega_l (m-1) K_{jl} (m) \tag{20}$$

so that summing over j and using $\sum_{j=1}^{F_m} K_{jl} (m) = 1$ yields

$$\sum_{j=1}^{F_m} \omega_j (m) \beta_{jl} (m) = \omega_l (m-1)$$

or in matrix notation

$$\omega (m-1) = \omega (m) B (m) \text{ with } \omega (0) = \omega \text{ and } p \equiv \omega (n) . \tag{21}$$

Here the $1 \times F_m$ vector $\omega (m)$ and $F_m \times F_{m-1}$ matrix $B (m)$ are given by $\omega (m) \equiv [\omega_j (m)], B (m) \equiv \left[\beta_{jl} (m)\right]$ with constant returns to scale implying that $B (m) \iota_{F_{m-1}} = \iota_{F_m}$. By repeated use of the relationship $\omega (m-1) = \omega (m) B (m)$ it then follows that $\omega = pB$ with

$$B \equiv B (n) B (n-1) \cdots B (1)$$

and where $B \iota_F = \iota_G$. The quantities of intermediate goods produced can be calculated from (20) as

$$K_{jl} (m) = \frac{\omega_j (m) \beta_{jl} (m)}{\omega_l (m-1)}.$$

10.2 Failure of the Regularity Assumptions

If C is decomposable then it is decomposable of some order $r > 1$ (see Grimmett and Stirzaker, 1982, p 123); that is we can partition the set of goods as if they come from $r+1$ islands as

$$\mathcal{G} = \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_r \cup \mathcal{G}_{r+1} \text{ where } \mathcal{G}_m \neq \varnothing \text{ for } m \neq r+1 \text{ and } \mathcal{G}_m \cap \mathcal{G}_n = \varnothing \text{ for } m \neq n$$

so that if we list the goods in $\mathcal{G}_1$ first, $\mathcal{G}_2$ second etc., then C takes the form

$$C = \begin{bmatrix} C_{11} & 0 & \cdots & 0 & 0 \\ 0 & C_{22} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{rr} & 0 \\ C_{1,r+1} & C_{2,r+1} & \cdots & C_{r,r+1} & C_{r+1,r+1} \end{bmatrix} \tag{22}$$

where C_{mm} is indecomposable for $m \neq r+1$. If $\mathcal{G}_{r+1} \neq \varnothing$ then all rows of $C_{r+1,r+1}$ sum to less than one. From this it follows that there are r basic solutions to $p^m = p^m C$ or $\tilde{p}^m C_{mm} = \tilde{p}^m$ for $m = 1, 2, \ldots, r$ where $p^m = \left[p_j^m\right]$ satisfies

$$p_j^m > 0 \text{ if and only if } j \in \mathcal{G}_m \text{ and } p^m \iota_G = \sum_{j \in \mathcal{G}_m} p_j^m = 1.$$

The general solution p is then any arbitrary convex combination of the basic solutions as

$$p_j = \sum_{m=1}^{r} \delta_m p_j^m \text{ for } \delta_m > 0 \text{ and } \sum_{n=1}^{r} \delta_m = 1.$$

Any goods $j \in \mathcal{G}_{r+1}$ in island $r+1$ have a zero price. The factors and households of island m are

$$\begin{aligned} \mathcal{F}_m &= \left\{l | \exists j \in \mathcal{G}_m \text{ such that } \beta_{jl} > 0\right\} \text{ for } m = 1, 2, \ldots, r \text{ and } \mathcal{F}_{r+1} = \{l | l \notin \mathcal{F}_m \text{ for } m \leq r\} \\ \mathcal{H}_m &= \{i | \exists l \in \mathcal{F}_m \text{ such that } \varepsilon_{li} > 0\} \text{ for } m = 1, 2, \ldots, r \text{ and } \mathcal{H}_{r+1} = \{i | i \notin \mathcal{H}_m \text{ for } m \leq r\} \end{aligned}$$

with basic solutions for factor prices and incomes for $m \neq r+1$ given by

$$\omega_l^m \equiv \sum_{j \in \mathcal{G}_m} p_j^m \beta_{jl} \text{ and } y_i^m \equiv \sum_{l \in \mathcal{F}_m} \omega_l^m \varepsilon_{li}.$$

From the definitions of $\mathcal{F}_m, \mathcal{H}_m$ and the fact that $p_j^m > 0$ if and only if $j \in \mathcal{G}_m$, it follows that $\omega_l^m > 0$ if and only if $l \in \mathcal{F}_m$, and $y_i^m > 0$ if and only if $i \in \mathcal{H}_m$. Since

$$\sum_{l=1}^{F} \omega_l^m = \sum_{i=1}^{H} y_i^m = 1$$

it follows that $\mathcal{F}_m \neq \varnothing$ and $\mathcal{H}_m \neq \varnothing$. For $j \notin \mathcal{G}_m$ we have

$$0 = p_j^m = \sum_{i \in \mathcal{H}_m} y_i^m \alpha_{ij}.$$

Since $y_i^m > 0$ for $i \in \mathcal{H}_m$ it follows that if $i \in \mathcal{H}_m$ and $j \notin \mathcal{G}_m$ then $\alpha_{ij} = 0$; that is island m residents do not demand non-island goods. For $i \notin H_m$ we have

$$y_i^m \equiv \sum_{l \in \mathcal{F}_m} \omega_l^m \varepsilon_{li} = 0.$$

Since $\omega_l^m > 0$ for $l \in \mathcal{F}_m$, it follows that if $l \in \mathcal{F}_m$ and $i \notin \mathcal{H}_m$ then $\varepsilon_{li} = 0$; that is non-island residents hold no island m factors. For $l \notin \mathcal{F}_m$ we have

$$\omega_l^m = \sum_{j \in \mathcal{G}_m} p_j^m \beta_{jl} = 0.$$

Since $p_j^m > 0$ for $j \in \mathcal{G}_m$ it follows that if $j \in \mathcal{G}_m$ and $l \notin \mathcal{F}_m$ then $\beta_{jl} = 0$; that is non-island factors are not used in the production of any island m goods.

From these results it follows that each island of factors, goods, and households $\mathcal{F}_m, \mathcal{G}_m, \mathcal{H}_m$ for $m = 1, 2, \ldots, r$ is self-contained with island $r+1$ consisting of households with zero incomes, unwanted goods, and unneeded factors. The general solution is

$$p_j = \sum_{m=1}^{r} \delta_m p_j^m,\ \omega_l = \sum_{m=1}^{r} \delta_m \omega_l^m, y_i = \sum_{m=1}^{r} \delta_m y_i^m \text{ for } \delta_m > 0 \text{ and } \sum_{m=1}^{r} \delta_m = 1$$

with a resulting allocation of resources given by

$$Q_{ij} = \frac{y_i \alpha_{ij}}{p_j}, K_{jl} = \frac{p_j \beta_{jl}}{\omega_l} \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}.$$

(Island $r+1$ is allocated no resources.) For $m \neq r+1$ with $i \in \mathcal{H}_m, j \in \mathcal{G}_m, l \in \mathcal{F}_m$ we have

$$p_j = \delta_m p_j^m,\ \omega_l = \delta_m \omega_l^m, y_i = \delta_m y_i^m$$

and so independent of δ we have

$$Q_{ij} = \frac{y_i^m \alpha_{ij}}{p_j^m} \text{ and } K_{jl} = \frac{p_j^m \beta_{jl}}{\omega_l^m}.$$

It follows that although there may be multiple solutions, the resulting allocation of resources is unique.

10.3 Firm Data

Suppose instead of industry data we are given an even more detailed set of data: the production and factor usage of each of N_j firms for each $j \in \mathcal{G}$ as

$$Q_j^+(n), K_{jl}^+(n) \text{ for } n = 1, 2, \ldots N_j$$

where we now define industry production and factor usage as

$$Q_j^+ \equiv \sum_{n=1}^{N_j} Q_j^+ (n), K_{jl}^+ \equiv \sum_{n=1}^{N_j} K_{jl}^+ (n) \text{ for all } j \in \mathcal{G}, l \in \mathcal{F}. \tag{23}$$

Instead of Data Assumption 4 we now adopt the zero profit assumption for each firm.

Data Assumption 4': *Each firm makes zero profits: For all* $j \in \mathcal{G}$

$$P_j^+ Q_j^+ (n) = \sum_{l=1}^{F} W_l^+ K_{jl}^+ (n) \text{ for } n = 1, 2, \ldots N_j. \tag{24}$$

This more detailed data set could falsify the fitted cGE model if there was some firm in some industry where the firm's factor share did not match the industry's factor share; that is if for some n, j, l it were the case that

$$\frac{W_l^+ K_{jl}^+ (n)}{P_j^+ Q_j^+ (n)} \neq \beta_{jl} \equiv \frac{W_l^+ K_{jl}^+}{P_j^+ Q_j^+}.$$

This can be remedied if we allow each of the N_j firms in industry j to have a different Cobb-Douglas production function. In particular we fit the same cGE model with exactly the same fitted matrices A, B, E, and with exactly the same predicted prices and incomes $p = p^+, y = y^+, \omega = \omega^+$; but where we assume firm n in industry j has a production function

$$Q_j (n) = \Gamma_j (n) \prod_{l=1}^{F} K_{jl} (n)^{\beta_{jl}(n)}$$

where

$$\beta_{jl} (n) = \frac{W_l^+ K_{jl}^+ (n)}{P_j^+ Q_j^+ (n)} \text{ and } \Gamma_j (n) = \frac{Q_j^+ (n)}{\prod_{l=1}^{F} K_{jl}^+ (n)^{\beta_{jl}(n)}}. \tag{25}$$

The fitted model has constant returns to scale since from Data Assumption 4' we have

$$\sum_{l=1}^{F} \beta_{jl} (n) = \frac{\sum_{l=1}^{F} W_l^+ K_{jl}^+ (n)}{P_j^+ Q_j^+ (n)} = 1.$$

Since each firm has constant returns to scale it will be indifferent to its scale of operation, and so we can assume that each firm produces the level of output $Q_j^+ (n)$ observed in the data. Profit maximization requires that $K_{jl} (n)$ satisfies

$$\frac{W_l^+ K_{jl}^+ (n)}{P_j^+ Q_j^+ (n)} \equiv \beta_{jl} (n) = \frac{W_l^+ K_{jl} (n)}{P_j^+ Q_j^+ (n)}$$

from which $K_{jl}(n) = K_{jl}^{+}(n)$ follows. This equilibrium is feasible since

$$\sum_{n=1}^{N_j} K_{jl}(n) = \sum_{n=1}^{N_j} K_{jl}^{+}(n) = E_l.$$

10.4 Solving $p = pC$ Without Matrix Inversion

It is possible to solve $p = pC$ exactly but without explicit matrix inversion. To begin assume that C is indecomposable and define $C^G \equiv C$ and $p^G \equiv p$. Given an $m \times m$ indecomposable stochastic matrix C^m with solution $p^m = p^m C^m$ and $p^m \iota_m = 1$, we reduce the dimension of C^m by one to C^{m-1} with solution $p^{m-1} = p^{m-1} C^{m-1}$ and $p^{m-1}\iota_{m-1} = 1$. This is done as follows: Partition C^m and p^m as

$$C^m = \begin{bmatrix} C_{11}^m & C_{12}^m \\ C_{21}^m & C_{22}^m \end{bmatrix} \text{ and } p^m = \begin{bmatrix} p_1^m & p_2^m \end{bmatrix}$$

where C_{22}^m and p_2^m are scalars. From $p^m = p^m C^m$ we have

$$p_1^m = p_1^m C_{11}^m + p_2^m C_{21}^m \text{ and } p_2^m = p_1^m C_{12}^m + p_2^m C_{22}^m.$$

Since C^m is indecomposable it follows that $C_{21}^m \neq 0$ and $C_{22}^m \neq 1$ so that $p_2^m = \frac{p_1^m C_{12}^m}{1 - C_{22}^m}$ and $p_1^m = p_1^m C^{m-1}$ where

$$C^{m-1} = C_{11}^m + \frac{C_{12}^m C_{21}^m}{1 - C_{22}^m}.$$

Here C^{m-1} is a $(m-1) \times (m-1)$ indecomposable stochastic matrix. That C^{m-1} is a stochastic matrix follows from $C^{m-1}\iota_{m-1} = \iota_{m-1}$, which follows from $C_{21}^m \iota_{m-1} = 1 - C_{22}^m$ and $C_{11}^m \iota_{m-1} + C_{12}^m = \iota_{m-1}$. That C^{m-1} is indecomposable follows from C^m being indecomposable. By induction then C^m is an indecomposable stochastic matrix for all $1 \leq m \leq G$. In each step of the computation of C^m we need only store C_{12}^m and C_{22}^m.

Since $p_1^m = p_1^m C^{m-1}$ it follows that p_1^m is proportional to p^{m-1}. Using $p^m \iota_m = 1$ we obtain

$$p^m = \begin{bmatrix} p_1^m & p_2^m \end{bmatrix} = \begin{bmatrix} \frac{p^{m-1}(1 - C_{22}^m)}{1 - C_{22}^m + p^{m-1} C_{12}^m} & \frac{p^{m-1} C_{12}^m}{1 - C_{22}^m + p^{m-1} C_{12}^m} \end{bmatrix}. \tag{26}$$

This process continues until we arrive at $C^1 = 1$ in which case $p^1 = 1$. Working backwards from $p^1 = 1$ using (26) we obtain the unique strictly positive solution to $p = pC$ and $p\iota_G = 1$.

If C is decomposable then we can modify the algorithm so that it still works. The problem will be that at some m we will find that either $C_{22}^m = 1$ or $C_{12}^m = 0$. First if $C_{12}^m = 0$ then C_{11}^m is a stochastic matrix and so we set $C^{m-1} = C_{11}^m$, and if required the algorithm continues. If $C_{12}^m = 0, C_{22}^m \neq 1$ then $p_2^m = 0$ and

$$p^m = \left[\begin{array}{cc} p^{m-1} & 0 \end{array} \right].$$

If $C_{12}^m = 0, C_{22}^m = 1$ then we have multiple solutions for p^m as

$$p^m = \left[\begin{array}{cc} (1-\delta)\, p^{m-1} & \delta \end{array} \right]$$

for any δ satisfying $0 < \delta < 1$. Since the resulting allocation does not depend on δ, we are free to pick any convenient value, say $\delta = \frac{1}{2}$. If $C_{22}^m = 1, C_{12}^m \neq 0$ then $p_1^m = 0$ and $p_2^m = 1$ so that the algorithm stops with

$$p^m = \left[\begin{array}{cc} p_1^m & p_2^m \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \end{array} \right].$$

In each of these three cases we as before work backwards from p^m using (26) to calculate p.