Time Series Econometrics and General Equilibrium

Michael Sampson
Department of Economics
Concordia University
1455 de Maisonneuve Blvd. W.
Montreal, Quebec
Canada H3G 1M8
Email: michael.sampson@concordia.ca

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Abstract

A *DGE* model is developed that has a solution for both log prices and quantities that is a standard time series econometric model. Definitions for globalization and convergence are given, and these are used to derive the time series implications for prices and quantities. In the applied work we find that the Grilli-Yang commodity price data set appears to be inconsistent with convergence and globalization.

1 Introduction

Standard time series econometrics fits the economic real world surprisingly well. Begin with just about any price or quantity. Fit a standard time series econometric (*TSE*) model, that is where the logarithm of the series is either difference or trend stationary. You can then be reasonably confident of success; that is the resulting model will explain a large part of the behavior of that time series. There may still remain effects in the data, such as *GARCH* or regime switching effects, but these too can be accomodated using modified versions of the basic *TSE* model.

The success of *TSE* models is largely statistical: there is no underlying economic model, and so there is no economics to guide econometric work, interpret results, or link hypotheses. Macroeconomics on the other hand begins with a Dynamic General Equilibrium (*DGE*) model, and so there is no problem with interpretation. Rather the problem stems from the success of *TSE* models. *DGE* models do not have a closed-form solution. It follows that the solution is *not* a *TSE* model, and hence *DGE* models predict the existence of effects *not* predicted by *TSE* models, just as general relativity predicts effects not predicted by Newton's law of gravity. But unlike general relativity where trans-Newtonian effects are actually observed, there is little evidence that any of the trans-*TSE* effects predicted by *DGE* models actually exist.

That *TSE* models are successful suggests that either 1) *DGE* models should have *TSE* solutions, or 2) that any predicted trans-*TSE* effects should be shown to actually exist in the data. In this paper we primarily deal with 1), although we do discuss 2) at the end of the paper. We consider a class of *DGE* models that have *TSE* solutions, which we call *cDGE* models. The *cDGE* model has a *TSE* solution for both prices and quantities, and is capable of generating both endogenous growth and endogenous seasonality. Prices and quantities can be either trend or difference stationary, and so the *cDGE* model makes it possible to give an economic interpretation to unit root tests. With the *cDGE* model it is possible to give a precise and testable definition of both globalization and convergence. We show that globalization implies that quantities are driven by a single trend, and this single trend must be either difference or trend stationary. Furthermore globalization implies convergence, which holds if and only if real prices are stationary. If globalization does not hold then quantities depend on $r > 1$ trends, some of which may be trend stationary while others may be difference stationary. We show that convergence can still hold even when globalization does not.

An advantage of the *cDGE* model is that it allows the use of price data to test convergence. This is an advantage because prices are much better measured and more widely available than the aggregate quantities (such as aggregate *GDP* per capita) traditionally used to test convergence.

In the applied part of the paper we use commodity price data to study convergence. While price data are generally better than quantity data, primary commodity price data are even better than other price data for studying convergence since, unlike say a car, the underlying commodity remains stable over long periods. Using the *cDGE* model we link the hypothesis of convergence with the Prebisch-Singer hypothesis in the commodity price literature; in particular if convergence holds then the Prebisch-Singer hypothesis must be false. Based on the empirical results the data appear to be inconsistent with both convergence and the Prebisch-Singer hypothesis.

We end the paper by considering what sort of effects in the data would lead one to reject the *cDGE* model. We show that the *cDGE* model is empirically equivalent to the *TSE* model. If the data are inconsistent with the *cDGE* model then it must be because the form of the non-stationarity is inconsistent with the *TSE* model. We illustrate how the *cDGE* model can be modified to take into account the existence of trans-*TSE* effects in the data, all the while preserving the existence of a closed-form solution.

2 The *cDGE* Model

Assume a perfectly competitive world economy made up of a set of n goods $N = \{1, 2, \ldots, n\}$. Let Q_{it} for $i \in N$ be the quantity of good i at time t. Time is discrete as $t = 0, \pm 1, \ldots$. Each good acts both as an input and an output in a Cobb-Douglas production function with constant returns to scale as

$$Q_{it} = \prod_{j=1}^{n} \left(Q_{jt-1}^{i}\right)^{a_{ij}} \exp\left(a_i + e_{it}\right) \text{ for all } i \in N \tag{1}$$

with $a_{ij} \geq 0$, $\sum_{j=1}^{n} a_{ij} = 1$, and where Q_{jt-1}^{i} is the amount of good j used in the production of good i at time t, with $\sum_{i=1}^{n} Q_{jt-1}^{i} = Q_{jt-1}$. Since there are constant returns to scale, the $n \times n$ matrix $A \equiv [a_{ij}]$ is a stochastic matrix (like a Markov chain) with rows that sum to one or

$$A\iota^n = \iota^n \tag{2}$$

where $\iota^n \equiv [1]$ is an $n \times 1$ vector of ones.

The $n \times 1$ vector of shocks $e_t \equiv [e_{it}]$ is assumed to be exogenous and stationary with Wold representation

$$e_t = \psi(L)\,\varepsilon_t$$

where $\psi(L) = \sum_{k=0}^{\infty} \psi_k L^k$ with $\psi_0 \equiv I$, $\varepsilon_t = \Omega z_t$ where the the $n \times 1$ vectors z_t are *i.i.d.* shocks with $E[z_t] = 0$ and $E\left[z_t z_t^T\right] = I$.[1] The variance-covariance matrix of ε_t is $\Sigma_\varepsilon \equiv \Omega\Omega^T$, which may or may not have full rank.

Later we will be using the Beveridge-Nelson (1981) decomposition as

$$e_t = \psi(L)\,\varepsilon_t = \psi(1)\,\varepsilon_t + (1-L)\,\bar{e}_t \text{ where } \bar{e}_t = \bar{\psi}(L)\,\varepsilon_t \text{ with } \bar{\psi}_k = -\sum_{j=k+1}^{\infty} \psi_j. \tag{3a}$$

Here $\bar{e}_t$ is assumed to be stationary, an assumption that will be satisfied under weak assumptions, for example if e_t has the short memory property, or if e_t is a stationary *VARMA(p,q)* process.

We assume a representative individual for the world economy with utility

$$U_t = E_t\left[\sum_{k=0}^{\infty} \beta^k \rho^T q_{t+k}\right] \tag{4}$$

where $0 < \beta < 1$ is the discount factor, where $q_t \equiv [q_{it}]$ with $q_{it} \equiv \ln(Q_{it})$ and $\rho = [\rho_i]$ are $n \times 1$ vectors with $\rho_i \geq 0$ and $\rho\iota^n = 1$.

In the Appendix we shown that the competitive equilibrium leads to time invariant input shares $\gamma_j^i \equiv \frac{Q_{jt}^i}{Q_{jt}}$ as

$$\gamma_j^i = \frac{d_i a_{ij}}{\sum_{k=1}^{n} d_k a_{kj}}$$

[1] For most of our results the z_t could be merely uncorrelated, instead of *i.i.d.*, so that, for example, z_t could have *ARCH* effects.

where the $1 \times n$ vector $d \equiv [d_i]$ is given by

$$d = (1 - \beta)\, \rho^T \left(I - \beta A\right)^{-1}.$$

The elements of d are non-negative and sum to one or $d\iota^n = 1$.

It follows that

$$q_{it} = b_i + \sum_{j=1}^{n} a_{ij} q_{jt-1} + e_{it} \text{ for all } i \in N$$

where

$$b_i = a_i + \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_j^i\right).$$

Defining the $n \times 1$ vector $b \equiv [b_i]$

$$q_t = b + Aq_{t-1} + e_t \tag{5}$$

and so all quantities in the world economy follow a *TSE* model.

In the Appendix it is shown that the competitive equilibrium can be supported by a sequence of spot markets with nominal prices given by

$$P_{it} = \frac{d_i Y_t}{Q_{it}} \text{ for } i = 1, 2 \ldots n \tag{6}$$

where $Y_t \equiv \sum_{i=1}^{n} P_{it} Q_{it}$. To construct real prices we use a geometric price index

$$P_t^* \equiv \prod_{i=1}^{n} P_{it}^{\theta_i} \tag{7}$$

where $\theta \equiv [\theta_i]$ is a $1 \times n$ row vector of weights with $\theta\iota^n = 1$. The use of a geometric price index insures that like quantities, real prices follow a *TSE* model.

Define $p_{it} \equiv \ln\left(\frac{P_{it}}{P_t^*}\right)$ as the log of the real price $\frac{P_{it}}{P_t^*}$, and let $p_t = [p_{it}]$ be the $n \times 1$ vector of the log of real prices. Since $(I - \iota^n \theta)\, \iota^n = 0$ it follows that

$$p_t = (I - \iota^n \theta)\, (\delta - q_t) \tag{8}$$

where the $n \times 1$ vector $\delta \equiv [\delta_i]$ is defined by $\delta_i \equiv \ln(d_i)$, and so prices follow a *TSE* model. Since $\theta\,(I - \iota^n \theta) = 0$, real prices satisfy the normalization

$$\theta p_t = 0.$$

3 Interpretations

The *cDGE* model is capable of an infinite number of interpretations depending on the number of goods n one assumes, and how one then interprets (gives names to) these n goods. For empirical work the exact interpretation is not important except with regards to the time series that are in the data set. For example in the econometric application below, we have data on 24 primary commodities, and so we are forced to commit to the interpretation that 24 of the n goods are these particular commodities. The remaining $n-24$ goods can be interpreted in any manner one pleases, and for any choice of $n \geq 24$.

We prefer to interpret many of the n goods as different forms of physical and human capital, labelled according to vintage, state of the world, and location. Thus in the production functions in (1) we have physical and human capital producing physical and human capital. If we think of people as collections of human capital, then each individual in the world is produced by a production function in (1). It may appear unrealistic in (1) that there is 100% depreciation, but this disappears if goods are appropriately labelled according to vintage. A 5-year-old car might be produced by combining a 4-year-old car with a 25-year-old mechanic's human capital, a garage, oil, and spark plugs. In the process a 26-year-old mechanic (a form of human capital) is produced by combining a 25-year-old mechanic with food, shelter, and a year of working on 4-year-old cars. Once the 5-year old-car and 26-year-old mechanic are produced there is 100% depreciation: nothing remains of the 4-year-old car and nothing remains of the 25-year old mechanic.

Storage technologies are also included in (1). A pure storage technology for good i, say iron in the ground or fish in the sea, would have $a_{ii} = 1, a_{ij} = 0$ for $i \neq j$ as

$$Q_{it} = Q^{i}_{it-1} e^{a_i + e_{it}}.$$

Other examples, say storing grain that has just been harvested, would require other inputs besides the good itself. Grain in the field is combined with transportation, a warehouse and warehouse workers, pest control, to produce a random amount of one-year-old wheat at that warehouse location. This one-year-old wheat could then be combined with transportation factors to produce one-year-old wheat in say a New York harbour.

Quantities in the *cDGE* model have a twofold nature: 1) they appear as inputs in the production function in (1) and so act as factors of production and 2) they appear in the utility function in (4) and so act as consumption goods. This again is realistic if unconventional: eating an apple may be pleasant, and hence an apple is a consumption good, but by keeping the doctor away this same apple maintains human capital as an input, just as oil keeps physical capital, say a motor, running. A computer can be used to write a report, but it can also be used to play a game or plan one's night out.

Because we are not forced into any choice for n or interpretation of the goods in N, the *cDGE* model can be realistic in a way that conventional *DGE* models cannot. That is, it is possible to imagine the production links of a *cDGE* model being as intricate and complex as the actual production links in the real world, but where we are not forced to actually list and name these links, which would be an impossible task. This is similar to Read's (1958) fable of a pencil that attempts to trace out its genealogy: the goods directly involved in its production, the goods involving the production of those goods, and so on, with the final conclusion that actually nobody knows how to make a pencil.

4 Convergence

The conventional notion of convergence, for example as found in Barro and Sala-i-Martin (1992), is defined in terms of per-capita aggregate output for countries or regions, and so as a matter of interpretation, one assumes a country or region that produces one homogeneous good with one kind of homogeneous labour, and that this good can either be consumed or transformed into capital. The conditions under such an interpretation would be valid are very restrictive and unlikely to hold. Furthermore It is questionable whether it is even meaningful to compare aggregate output measures across different countries, regions, and time periods. As well the required data are often lacking or of poor quality, especially as one goes further back in time or looks at less developed countries or regions. This is troublesome since it is precisely over long periods and for poorer countries and regions that the hypothesis of convergence is the most interesting.

We propose an alternative definition of convergence that is based on individual quantities, and so does not suffer from these problems: convergence holds if *all* quantities in the world economy share a common scalar trend τ_t.

Definition 1 ***Convergence*** *holds if*

$$q_t = \bar{q}_t + \iota^n \tau_t \text{ where } \tau_t = \mu + \tau_{t-1} + \kappa_t \tag{9}$$

where the trend τ_t is a scalar, and where the $n \times 1$ vector $\bar{q}_t$ and scalar κ_t are stationary with $E[\kappa_t] = 0$.

This definition is dependent on any interpretation of the quantities in the model since it is defined in terms of individual goods, and so does not require any aggregation assumptions.

If convergence holds then $E[\Delta q_t] = \iota^n \mu$, and so all quantities grow at a rate μ, determined by the common trend τ_t. We will see that τ_t is either difference or trend stationary. Thus if convergence holds it must be that all quantities are all either trend stationary or difference stationary (it is impossible for some to be trend stationary and some to be difference stationary).

If in our interpretation we imagine a number of countries or regions, with each country or region specializing in some subset of the goods in N; then if convergence holds each country or region will grow at the common rate μ independent of the goods it specializes in.

From (8) and $(I - \iota^n \theta)\, \iota^n = 0$ we have that if convergence holds then

$$p_t = (I - \iota^n \theta)(\delta - \bar{q}_t)$$

and so real prices are stationary.

Theorem 2 *If convergence holds then p_t is stationary.*

Later we will see that convergence is both necessary and sufficient for prices to be stationary. Thus it is possible to study convergence indirectly using price data, which is typically better measured and much more available than quantity data.

5 Decomposability and Globalization

The time series properties of q_t and p_t depend on the eigenvalues of the matrix A. Since A is a stochastic matrix, by the Perron-Frobenius theorem (see Grimmett and Stirzaker (1982)), the matrix A has an eigenvalue $\lambda^* = 1$ of maximum modulus; that is if λ_k is any other eigenvalue of A then $|\lambda_k| \leq 1$. The eigenvalues where $|\lambda_k| < 1$ result in stationary dynamics, while the eigenvalues with $|\lambda_k| = 1$ result in non-stationary dynamics. Eigenvalues where $|\lambda_k| = 1$ but $\lambda_k \neq 1$ are only possible when A has a special form, A is cyclic, and this results in non-stationary seasonality. This case would be of interest in a theory of endogenous seasonality, but since this is not the focus of this paper we assume A is acyclic so that $|\lambda_k| = 1$ implies that $\lambda_k = 1$.

The important question for our purposes is whether $\lambda^* = 1$ is unique or not; that is whether A is decomposable or indecomposable. The matrix A is decomposable if there is a partition of N as $N = N_0 \cup N_1$ such that if we list the goods for which $i \in N_0$ first, and the goods for which $i \in N_1$ second, then A takes the form

$$A = \begin{bmatrix} A_0^0 & A_0^1 \\ 0 & A_1^1 \end{bmatrix}. \tag{10}$$

If no such partition exists then A is indecomposable, or we say A is decomposable of order $r = 0$. If A is decomposable then, as we will see, it is decomposable of some integer order $r > 0$.

If A is indecomposable then the production of each good in N requires, either directly or indirectly, all other inputs in N. To put this more precisely, we say good $j_2 \in N$ is directly linked to good $j_1 \in N$, written as $j_2 \Longrightarrow j_1$, if j_1 appears in the production function of j_2, or $a_{j_2 j_1} > 0$. We say that good j_1 is linked (either directly or indirectly) to good j_n, written as $j_1 \rightarrow j_n$, if there exists a sequence $j_2, \ldots, j_{n-1} \in \mathcal{G}$ such that

$$j_1 \Longrightarrow j_2 \Longrightarrow \cdots \Longrightarrow j_n.$$

The fact that A is indecomposable then means that

$$j_1 \rightarrow j_2 \text{ for all } j_1, j_2 \in N.$$

That is all goods in the economy are ultimately involved in the production of a pencil, and a pencil is ultimately involved in the production of all goods in the economy, including pencils. Mathematically this means that all elements of the matrix A^n are strictly positive.

We can thus interpret an indecomposable A as implying that the world economy is characterized by *globalization in production* in that all goods are either directly or indirectly produced by all other goods.

Suppose then that A is indecomposable. By the Perron-Frobenius theorem $\lambda^* = 1$ is the unique eigenvalue of modulus one, and associated with $\lambda^* = 1$ is a unique $1 \times n$ eigenvector $v \equiv [v_i]$ with $v_i > 0$ for all i satisfying

$$vA = v \text{ and } v\iota^n = 1. \tag{11}$$

Multiplying both sides of (5) by v we obtain the scalar trend $\tau_t \equiv vq_t$ for q_t as

$$\tau_t = \mu + \tau_{t-1} + ve_t \tag{12}$$

where ve_t is stationary and the growth rate of the trend is $\mu \equiv vb$. Multiplying both sides of (5) by $I - \iota^n v$, defining

$$\bar{q}_t \equiv (I - \iota^n v)\, q_t \tag{13}$$

and $\bar{A} \equiv (I - \iota^n v)\, A$, and using

$$\bar{A} = A\,(I - \iota^n v) = \bar{A}\,(I - \iota^n v)$$

yields

$$\bar{q}_t = (I - \iota^n v)\, b + \bar{A}\bar{q}_{t-1} + (I - \iota^n v)\, e_t. \tag{14}$$

Since ι^n and v are the right and left-hand eigenvectors of A corresponding to $\lambda^* = 1$, the eigenvalues of $\bar{A}$ are identical to those of A except that $\lambda^* = 1$ is replaced by an eigenvalue of 0. Since all eigenvalues of $\bar{A}$ are strictly less than 1 in absolute value it follows that $\bar{q}_t$ is stationary. Since

$$\bar{q}_t \equiv (I - \iota^n v)\, q_t = q_t - \iota^n \tau_t$$

we have

$$q_t = \bar{q}_t + \iota^n \tau_t$$

and so convergence holds as defined by (9). Since the shocks e_t are stationary and all quantities grow at a common rate μ, we have a model of endogenous growth.

Theorem 3 *If A is indecomposable then $q_t = \bar{q}_t + \iota^n \tau_t$ and convergence holds.*

From Theorem 2 we then have the following result:

Theorem 4 *If A is indecomposable then p_t is stationary.*

Given its form the scalar trend τ_t will be either difference or trend stationary, and accordingly all components of q_t will be all either difference or trend stationary. From (12) and (3a) we have

$$(1 - L)\,\tau_t = \mu + v\psi\,(1)\,\Omega z_t + (1 - L)\, v\bar{e}_t$$

and so multiplying both sides by

$$\frac{1 - L^t}{1 - L} = 1 + L + L^2 + \cdots L^{t-1}$$

yields

$$\tau_t = \bar{\tau}_t + \mu t + v\psi(1)\Omega \sum_{j=1}^{t} z_j \tag{15}$$

where $\bar{\tau}_t = \tau_0 + v(\bar{e}_t - \bar{e}_0)$ is stationary. The term $\sum_{j=1}^{t} z_j$ is an $n \times 1$ vector of pure random walks. Therefore whether τ_t is trend or difference stationary depends on $v\psi(1)\Omega$.

If $v\psi(1)\Omega \neq 0$ then τ_t has a unit root, so that both τ_t and q_t are difference stationary. Since all quantities in q_t share the common scalar trend τ_t, q_t is cointegrated with $n-1$ linearly independent cointegrating vectors c^i satisfying $c^i \iota^n = 0$.

If $v\psi(1)\Omega = 0$ then $\tau_t = \bar{\tau}_t + \mu t$ and τ_t has no random walk component, so that both τ_t and q_t are trend stationary. Since the vector v is strictly positive, this can only occur if either $\psi(1)$ or Ω has a rank less than n, and at least one of the elements of either $\psi(1)$ or Ω is negative.

Without a more explicit interpretation of the model it is difficult to think of a good economic reason why $v\psi(1)\Omega = 0$ should hold exactly, and so the natural presumption would be that τ_t and q_t will be difference stationary. At the same time there is no natural reason to presume that $v\psi(1)\Omega$ is large either, and so the random walk component could be so small that for all practical purposes τ_t and q_t are trend stationary.

To summarize, if A is indecomposable we have globalization in production, convergence, and prices are stationary. All goods share the same scalar trend τ_t, and so all goods are either difference or trend stationary depending on $v\psi(1)\Omega$.

5.1 Globalization

Convergence can still hold if A is decomposable. Consider the case where A is decomposable of order $r = 1$; this means we can partition the goods into two non-empty groups N_0, N_1 so that A can be written as in (10) with $n_0 > 0$, A_1^1 indecomposable, and $A_0^1 \neq 0$. We then define globalization as follows.

Definition 5 ***Globalization:*** *The world economy is globalized if A is decomposable of order $r \leq 1$.*

If A is decomposable of order $r = 1$ then we can partition q_t, b, e_t accordingly as

$$q_t^T = \begin{bmatrix} (q_t^0)^T & (q_t^1)^T \end{bmatrix}, b^T = \begin{bmatrix} (b^0)^T & (b^1)^T \end{bmatrix}, e_t^T = \begin{bmatrix} (e_t^0)^T & (e_t^1)^T \end{bmatrix}$$

so that

$$\begin{aligned} q_t^0 &= b^0 + A_0^0 q_{t-1}^0 + A_0^1 q_{t-1}^1 + e_t^0 \\ q_t^1 &= b^1 + A_1^1 q_{t-1}^1 + e_t^1. \end{aligned}$$

The goods in N_0 use both goods in N_0 and N_1 as inputs, but they are not inputs for goods in N_1. We therefore interpret goods in N_0 as *pure consumption goods.*

Although strictly speaking goods in N_0 can act as inputs, this is only for goods in N_0. Thus a pure consumption good in N_0 might be a banana cream pie that requires, in addition to physical and human capital, bananas and cream.

Since A_1^1 is indecomposable Theorem 3 applies to q_t^1 above, and so we have

$$q_t^1 = \bar{q}_t^1 + \iota^{n_1} \tau_t$$

where $\bar{q}_t^1$ is stationary and the trend $\tau_t = v^1 q_t^1$ satisfies

$$\tau_t = \mu_1 + \tau_{t-1} + v^1 e_t^1$$

where the strictly positive $1 \times n_1$ eigenvector v^1 satisfies $v^1 A_1^1 = A_1^1$ and $v^1 \iota^{n_1} = 1$, and the growth rate is $\mu_1 = v^1 b^1$.

Now consider the goods in N_0. Since $A_0^1 \neq 0$ the eigenvalues of A_0^0 are strictly less than one in absolute value, and so

$$q_t^0 = \left(I - A_0^0\right)^{-1} b^0 + \left(I - A_0^0 L\right)^{-1} A_0^1 L q_t^1 + \left(I - A_0^0 L\right)^{-1} e_t^0.$$

Using $q_t^1 = \bar{q}_t^1 + \iota^{n_1} \tau_t$ in this expression, collecting all the stationary terms in $\bar{q}_t^0$, and using the fact that

$$\left(I - A_0^0\right)^{-1} A_0^1 \iota^{n_1} = \iota^{n_0} \text{ since } A_0^0 \iota^{n_0} + A_0^1 \iota^{n_1} = \iota^{n_0}$$

yields

$$q_t^0 = \bar{q}_t^0 + \iota^{n_0} \tau_t$$

where $\bar{q}_t^0$ is stationary. Combining this $q_t^1 = \bar{q}_t^1 + \iota^{n_1} \tau_t$ yields

$$q_t = \bar{q}_t + \iota^n \tau_t$$

and so if A is decomposable of order $r = 1$ then convergence holds and prices p_t are stationary. The only difference from the case where A is indecomposable as in Theorem 3 is that the trend τ_t is determined solely by the goods in N_1; the pure consumption goods in N_0 play no role.

We can then summarize our results so far as follows.

Theorem 6 ***Implications of Globalization:*** *If the world economy is globalized then convergence holds and prices are stationary.*

5.2 Production Islands

The next question then is if globalization does not hold, does convergence hold? We will now that generally the answer is no, but that it is not impossible for convergence to hold if certain restrictions hold.

If A is decomposable of order $r > 1$; then (see Grimmett and Stirzaker (1982)) there is a partition of N as

$$N = N_0 \cup N_1 \cup \cdots \cup N_r$$

with n_j the number of unique elements in N_j, such that A can be written as

$$A = \begin{bmatrix} A_0^0 & A_0^1 & A_0^2 & \cdots & A_0^r \\ 0 & A_1^1 & 0 & \cdots & 0 \\ 0 & 0 & A_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_r^r \end{bmatrix} \tag{16}$$

where the $n_j \times n_j$ matrices A_j^j are indecomposable and $n_j > 0$ for $j \geq 1$. Here there are r production islands N_j for $j = 1, 2, \ldots, r$. As before we interpret the goods in N_0 as pure consumption goods. It is possible that there are no pure consumption goods, or $n_0 = 0$, in which case A would be block diagonal. If $n_0 > 0$ then the row sums of A_0^0 are all strictly less than 1, so that each good in N_0 uses an input from at least one of the r production islands. (In the theory of Markov chains the states in N_0 are transitory states; the states in N_j for $j \geq 1$ are absorbing states.)

Define the $n_j \times 1$ vectors $q_t^j = [q_{it}], b^j = [b_i], e_t^j = [e_{it}]$ for $i \in N_j$. For the r production islands we have

$$q_t^j = b^j + A_j^j q_{t-1}^j + e_t^j \text{ for } j = 1, 2, \ldots, r.$$

Since A_j^j is indecomposable Theorem 3 applies and so

$$q_t^j = \bar{q}_t^j + \iota^{n_j} \tau_t^j \text{ where } \tau_t^j = \mu_j + \tau_{t-1}^j + v^j e_t^j$$

where $\bar{q}_t^j$ is stationary, $\tau_t^j = v^j q_t^j$, $\mu_j = v^j b^j$ with $v^j A_j^j = v^j$ and $v^j \iota^{n_j} = 1$. Thus the world economy is characterized by r trends τ_t^j each with a growth rate μ_j for $j = 1, 2, \ldots, r$.

For the pure consumption goods in N_0, assuming $n_0 > 0$, we have

$$q_t^0 = b^0 + A_0^0 q_{t-1}^0 + \sum_{j=1}^{r} A_0^j L q_t^j + e_t^0.$$

Substituting $q_t^j = \bar{q}_t^j + \iota^{n_j} \tau_t^j$ for $j = 1, 2, \ldots, r$ into this expression, collecting the stationary terms in $\bar{q}_t^0$, and defining the non-negative $n_0 \times 1$ vectors

$$\sigma^j \equiv \left(I - A_0^0\right)^{-1} A_0^j \iota^{n_j} \text{ for } j = 1, 2, \ldots, r$$

yields

$$q_t^0 = \bar{q}_t^0 + \sum_{j=1}^{r} \sigma^j \tau_t^j.$$

These vectors satisfy $\sum_{j=1}^{r} \sigma^j = \iota^{n_0}$ since

$$A_0^0 \iota^{n_0} + \sum_{j=1}^{r} A_0^j \iota^{n_j} = \iota^{n_0}.$$

Thus q_t^0 is a convex combination of the r trends coming from the r production islands.

If globalization does not hold so that $r > 1$, then the world economy has r trends, and this will generally mean that convergence will not hold. However it is possible the r trends are linearly dependent in such a way that they are all driven by a common scalar trend τ_t, in which case convergence would hold. To investigate this issue we need to consider the world economy as one system. To this end define the $n \times r$ matrix B as

$$B = \begin{bmatrix} \sigma^1 & \sigma^2 & \cdots & \sigma^r \\ \iota^{n_1} & 0 & \cdots & 0 \\ 0 & \iota^{n_2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \iota^{n_r} \end{bmatrix}.$$

The matrix B has non-negative elements, rows that sum to 1 or $B\iota^r = \iota^n$, and a rank of r. If x is any $r \times 1$ vector and α is any scalar then

$$Bx = \iota^n \alpha \Longleftrightarrow x = \iota^r \alpha. \tag{17}$$

Using the matrix B we are able to write the world economy as one system as

$$q_t = \bar{q}_t + BT_t \tag{18}$$

where $\bar{q}_t$ is stationary and T_t is an $r \times 1$ vector of trends with

$$T_t = M + T_{t-1} + Ve_t$$

where $M = \left[\mu_j\right]$ is the $r \times 1$ vector of the growth rates for each production island, $V \equiv \left[V^j\right]$ is the $r \times N$ matrix of $1 \times n$ eigenvectors V^j satisfying $V^j A = V^j$ where $V^j \iota^n = 1$ (where $V_i^j = v_i^j$ for $i \in N_j$ and $V_i^j = 0$ for $i \notin N_j$). Using the Beveridge-Nelson decomposition on T_t we have

$$T_t = \bar{T}_t + Mt + V\psi\left(1\right)\Omega \sum_{j=1}^{t} z_j \tag{19}$$

where $\bar{T}_t$ is stationary. For convergence to hold all trends must have the same growth rate and there must be at most one common scalar random walk so

$$M = \iota^r \mu \text{ and } V\psi\left(1\right)\Omega = \iota^r \phi \tag{20}$$

where ϕ is a $1 \times n$ vector. If (20) holds then

$$T_t = \bar{T}_t + \iota^r \tau_t$$

where $\tau_t = \mu + \tau_{t-1} + \phi z_t$ is a scalar random walk. From (18) we then have

$$q_t = \bar{q}_t + BT_t = \bar{q}_t + B\bar{T}_t + \iota^r \tau_t$$

where $\bar{q}_t + B\bar{T}_t$ is stationary, and hence convergence holds.

5.3 The Behavior of Prices

We now investigate the properties of prices when globalization does not hold and A is decomposable of order $r > 1$. From (18) we have

$$p_t = (I - \iota^n \theta)\, q_t = \bar{p}_t - (I - \iota^n \theta)\, B T_t$$

where $\bar{p}_t = (I - \iota^n \theta)\, \bar{q}_t$ is stationary. Here the r trends in T_t get filtered through the matrix $(I - \iota^n \theta)\, B$. Since B has a rank of r, the matrix $(I - \iota^n \theta)\, B$ has a rank $r - 1$ since from (17) we have

$$(I - \iota^n \theta)\, Bx = 0 \Longrightarrow Bx = \iota^n \theta Bx \Longrightarrow x = \iota^n \alpha.$$

In general then p_t will be driven by $r - 1$ linearly independent trends, and so p_t will be non-stationary if $r > 1$. However if (20) holds then convergence holds, in which case p_t will be stationary. The converse also holds: if p_t is stationary then (20) holds.

Theorem 7 *Convergence holds if and only if p_t is stationary.*

Proof. If A is decomposable of order $r > 1$ we have

$$p_t = (I - \iota^n \theta) \left(\delta - \bar{q}_t - B\bar{T}_t\right) - (I - \iota^n \theta)\, BM \times t - (I - \iota^n \theta)\, BV\psi\,(1)\,\Omega \sum_{j=1}^{t} z_j.$$

If p_t is stationary then

$$(I - \iota^n \theta)\, BM = 0 \text{ and } (I - \iota^n \theta)\, BV\psi\,(1)\,\Omega = 0$$

so

$$BM = \iota^n \sigma BM \text{ and } BV\psi\,(1)\,\Omega = \iota^n \sigma BV\psi\,(1)\,\Omega.$$

From (17) it then follows that $M = \iota^r \mu$ and $V\psi\,(1)\,\Omega = \iota^r \phi$ where $\mu = \sigma BM$ and $\phi = \sigma BV\psi\,(1)\,\Omega$. Hence the condition in (20) holds. ■

Thus from Theorem 7 we are able to study convergence indirectly using prices. In particular any finding of non-stationarity in real prices is inconsistent with the hypothesis of convergence.

6 An Empirical Application

In this section we use commodity price data to test for convergence and globalization. We use Theorem 7 to test convergence by examining the stationarity of the real price series. Our data set is the Grilli-Yang (1988) data: the annual prices of 24 primary commodities from 1900 to 2007, taken from Pfaffenzellar, Newbold, and Rayner (2007). Compared to high frequency financial data this is a small number of observations, but it has a long span: 108 years. Having a long span is more important when dealing with such long-run hypotheses convergence, trends and unit roots.

Price data is often better than quantity data; but commodity prices are particularly good. Unlike most goods, the fundamental nature of commodities remains constant. A car in the year 2000 hardly the same thing as a car in the year 1940, and cars hardly existed in the year 1900. But copper or wheat in the year 2000 is the same as copper or wheat in the year 1900. So in this sense commodity prices are ideal for studying long-run hypotheses.

Furthermore commodity prices are more relevant for developing regions and countries, which often rely on commodities for their export earnings, and where the concept of convergence is of more interest (see Deaton (1999), von Hagen (1988), and Hadass and Williamson (2003)).

There exists an extensive empirical literature on commodity prices, much of which centers on the Prebisch-Singer hypothesis (see Prebisch (1959) and Singer (1950)). Prebisch and Singer hypothesized that real primary commodity prices, relative to manufactured goods index, are trend stationary with a negative trend. The Prebisch-Singer hypothesis has been used to argue that Third World countries should ignore any comparative advantage in primary commodities and instead promote home grown industrialization through import substitution (von Hagen (1988)).

However, the reasons given for why the Prebisch-Singer hypothesis might hold are vague at best, a point made by both Deaton (1999) and Hadass and Williamson (2003). Singer's argument was based on technological progress augmenting the demand for manufactured goods more than raw materials, as well as a low income elasticity for food. Prebisch's argument was based on the Keynesian economics of the time.

Using the *cDGE* model we are able to interpret the conditions under which the Prebisch-Singer hypothesis holds. First since Theorem 7 requires real prices to be stationary if convergence and globalization hold, it follows that convergence and globalization are inconsistent with the Prebisch-Singer hypothesis. If the Prebisch-Singer hypothesis is true it must be possible to break down world production into a set of non-interacting production islands with some of those islands having manufactured goods, while others have the various commodities. Furthermore since Prebisch-Singer hypothesis requires that commodity prices have a negative trend, it must be that the production islands that these commodities belong to must have a *higher* growth rate than the production islands producing manufactured goods. That is the production islands producing commodities must be *more* efficient than the production islands producing manufactured goods, something that is, we think, contrary to the vague intuition behind the Prebisch-Singer hypothesis found in the literature.

Lutz (1999) provides a nice summary of the empirical literature on commodity prices up to 1999, and concludes that the data appears to support a difference stationary version of the Prebisch-Singer hypothesis with possible breaks in the trend in 1920 or 1921 or 1974. Kim et al. (2003) find only modest evidence for trends in 8 of the 24 commodities in the Grilli-Yang data set, with only 6 having negative trends. Kellard and Wohar (2006)) find that 10 commodity prices in the Grilli-Yang data set are difference stationary, 10 are trend stationary with 2 breaks, and 4 are trend stationary with one break, with little support for the

Prebisch-Singer hypothesis.

One of the advantages our study has compared to this literature is that the *cDGE* model tells us the required form for the price index used: it must be a geometric price index as in (7). We use the geometric price index for manufacturing goods found in Pfaffenzellar, Newbold, and Rayner (2007).

For each real commodity price p_{it} so constructed we assume that

$$p_{it} = \delta_0 + \delta_1 t + \sum_{j=1}^{p} \phi_j p_{it-j} + a_{it}.$$

Based on an examination of the partial autocorrelation functions for each of the 24 series it was decided to use $p = 5$ for each series. It is convenient to transform this model into an equivalent form given by the familiar augmented Dickey-Fuller (*ADF*) regression

$$\Delta p_{it} = \delta_0 + \delta_1 t + \delta_2 p_{it-1} + \sum_{j=1}^{p-1} \theta_j \Delta p_{it-j} + a_{it} \qquad (21)$$

where $\delta_2 \equiv \sum_{j=1}^{p} \phi_j - 1$. Under the null hypothesis that convergence and globalization hold, it follows that p_{it} is stationary and so we can test the restrictions $\delta_1 = 0$ and $\delta_2 < 0$.

The simplest test of convergence and globalization is to estimate (21) and test for the existence of a trend, or $H_o : \delta_1 = 0$, using a conventional t test. This t statistic and significance level are shown for each commodity in the third and fourth columns in Table 1 below. From Table 1 we see that most of the 24 commodities have a statistically significant trend, and this is inconsistent with both convergence and globalization.

Next we examine whether prices appear to be difference or trend stationary. First we make difference stationarity the null hypothesis and test $H_o : \delta_2 = 0$ against $H_1 : \delta_2 < 0$ using a standard *ADF* test. Second we make trend stationarity the null hypothesis and test trend stationarity against the alternative of difference stationarity using the *KPSS* test from Kwiatkowski et al. (1992). The results of both of these tests are shown below in Table 1 along with the corresponding significance levels.[2]

[2] Table 1 Significance Levels: $* = 10\% - 5\%$, $** = 5\% - 2.5\%$, $*** = 2.5\% - 1\%$, $**** = 1\% - 0\%$. DS = Difference Stationary, TS+ = Trend Stationary with a Positive Trend, TS- = Trend Stationary with a Negative Trend.

Series	Growth Rate (TS)	t Statistic	Significance	Growth Rate (DS)	t Statistic	ADF	Significance	KPSS	Significance	Conclusion
Beef	0.021	2.9	****	0.015	0.7	-3.35	*	0.10		TS +
Lamb	0.021	2.8	****	0.018	0.7	-3.39	**	0.10		TS +
Timber	0.013	3.9	****	0.014	1.2	-4.25	****	0.10		TS +
Wheat	-0.005	-3.2	****	-0.003	-0.4	-3.85	***	0.11		TS -
Sugar	-0.007	-2.4	***	-0.009	-0.6	-3.71	***	0.09		TS -
Rice	-0.009	-3.3	****	-0.008	-0.8	-3.58	***	0.12	*	TS-
Palm Oil	-0.008	-3.3	****	-0.005	-0.5	-3.70	***	0.24	****	DS or TS
Silver	0.013	2.1	**	0.006	0.5	-2.01		0.34	****	DS
Tobbaco	0.009	1.5		0.011	1.0	-2.41		0.27	****	DS
Copper	0.006	1.7	*	0.006	0.5	-1.99		0.26	****	DS
Cocoa	0.006	1.2		-0.006	-0.4	-2.56		0.20	***	DS
Tin	0.006	1.6		0.007	0.6	-2.71		0.17	**	DS
Zinc	0.006	2.7	****	0.007	0.7	-3.01		0.27	****	DS
Coffee	0.004	1.1		0.003	0.2	-2.93		0.18	***	DS
Banana	0.000	0.1		0.003	0.3	-2.35		0.20	***	DS
Lead	0.000	0.2		0.007	0.5	-2.56		0.17	**	DS
Tea	-0.003	-0.7		-0.005	-0.5	-2.21		0.25	****	DS
Hides	-0.004	-0.9		-0.003	-0.3	-2.18		0.31	****	DS
Jute	-0.006	-2.0	**	-0.002	-0.2	-2.46		0.32	****	DS
Aluminium	-0.006	-0.9		-0.012	-0.8	-2.23		0.33	****	DS
Maize	-0.008	-3.0	****	-0.005	-0.7	-3.09		0.31	****	DS
Cotton	-0.013	-2.5	***	-0.010	-1.2	-1.98		0.37	****	DS
Wool	-0.016	-2.1	**	-0.007	-0.8	-2.12		0.37	****	DS
Rubber	-0.016	-1.6		-0.019	-1.0	-2.51		0.29	****	DS

Based on these results it appears that beef, lamb, timber, wheat, sugar, rice and perhaps palm oil are trend stationary. The model predicts that commodities with negative trends (wheat, sugar, and rice) come from production islands with above average rates of growth, and commodities with positive trends (beef, lamb, and timber) come from production islands with below average rates of growth. The remaining commodities all appear to be difference stationary.

From these results none of the commodity prices appear to be stationary. Thus from Theorem 7 we are lead to reject convergence and globalization. That is the world economy can be broken down into $r > 1$ production islands. The fact that some commodity prices appear to be difference stationary while others appear to be trend stationary is consistent with having $r > 1$, where some island economies have a unit root while others do not. Alternatively the coefficients on the random walk terms in the trends driving these prices, as given by $V\psi(1)\Omega$ in (3a), may be too small to allow us to reject trend stationarity.

Another interpretation of these results is that convergence and globalization really do hold, but that there are near-islands in the world economy where the production links between these near-islands exist but are very weak. The apparent non-stationarity of real commodity prices is then really stationary behavior with so much persistence that it might take over 100 years, the approximate length of our sample, for the series to return to their long-run values.

7 Beyond the *cDGE* Model

Throughout this paper we have used the *cDGE* model as a maintained hypothesis. Under what circumstances would we reject the *cDGE* model? Below we will show that if we only have price data or quantity data, then the empirical validity of the *cDGE* model is equivalent to the empirical validity of the *TSE* model. The *TSE* model, and hence the *cDGE* model, is consistent with only three forms of non-stationary: seasonality, unit roots, and linear trends, and so it is here one must search to find effects that would invalidate the *cDGE* model.

We first show that the *cDGE* model places *no* restrictions on the form of the stationarity, and so the *cDGE* and *TSE* models are equivalent. Suppose the matrix A is indecomposable in

$$q_t = b + Aq_{t-1} + e_t$$

and suppose that

$$e_t = \left(I - \bar{A}L\right) x_t \text{ and } b = \left(I - \bar{A}\right) \tilde{b}$$

where x_t is *any* stationary process with $E\left[x_t\right] = 0$, and $\tilde{b}$ is an arbitrary $n \times 1$ vector of constants. Then

$$\left(I - AL\right) q_t = \left(I - \bar{A}\right) \tilde{b} + \left(I - \bar{A}L\right) x_t \text{ or } \left(I - \bar{A}L\right)^{-1} \left(I - AL\right) q_t = \tilde{b} + x_t.$$

Now

$$\left(I - \bar{A}L\right)^{-1} \left(I - AL\right) = \left(I - \bar{A}L\right)^{-1} \left(I - \bar{A}L - \iota^n vL\right) = I - \iota^n vL$$

and so

$$q_t = \iota^n \tau_{t-1} + \tilde{b} + x_t \tag{22}$$

so that the stationary dynamics of q_t, as given by $\tilde{b} + x_t$, can be any stationary process.[3] A similar argument holds if we have price data. From (22) we have

$$p_t = \left(I - \iota^n \theta\right) \left(\delta - q_t\right) = \left(I - \iota^n \theta\right) \left(\delta - \tilde{b} - x_t\right).$$

If we set the arbitrary constant term as $\tilde{b} = \delta - \tilde{\delta}$ where $\tilde{\delta}$ is arbitrary, then

$$p_t = \left(I - \iota^n \theta\right) \left(\delta - q_t\right) = \left(I - \iota^n \theta\right) \left(\tilde{\delta} - x_t\right)$$

and so aside from satisfying the normalization $\theta p_t = 0$, the model puts no restrictions on the stationary dynamics of p_t.

What is testable is the form of the non-stationarity predicted by the *cDGE* model. The *TSE* and hence *cDGE* model is only consistent with three forms of non-stationary: seasonality, unit roots, and linear trends. So for example a quadratic trend would be inconsistent with the *cDGE* model, even though there appears to be little evidence for this. A more relevant possibility, especially in the commodity price literature, is an exogenous break in a linear trend, which

[3] In the non-stationary trend component of q_t, as given above by $\tau_{t-1} = vq_{t-1}$, the vector v comes from the matrix A as $vA = v$ with $v\iota^n = 1$, and so one might hope that there is something testable here. But let $\tilde{v}$ be any $1 \times n$ vector that satisfies $\tilde{v}\iota^n = 1$ and let $\tilde{\tau}_t \equiv \tilde{v}q_t$. Then from the representation $q_t = \iota^n \tau_t + \bar{q}_t$ we have

$$\tilde{\tau}_t \equiv \tilde{v}q_t = \tau_t + \tilde{v}\bar{q}_t$$

and so $\tilde{\tau}_t$ differs from τ_t only by the stationary term $\tilde{v}\bar{q}_t$. That is there is nothing testable coming from the presence of v in the trend term either.

has been found in more than one study (see Lutz (1999) and Kellard and Wohar (2006)). As well the data can be inconsistent with the *cDGE* model if we have *both* price and quantity data. If such data exist then it is possible to test the constant budget share property of Cobb-Douglas in (6) that

$$\frac{P_{it}Q_{it}}{Y_{it}} = d_i \text{ for } i = 1, 2, \ldots, n.$$

Suppose then that from the data we are led to reject the *cDGE* model. It does not follow that the next step should be to entertain a *DGE* model without a closed-form-solution. In what follows we illustrate how it might be possible to amend the *cDGE* model in such a way that there is still a closed-form solution.

In the first we introduce demand shocks by assuming that utility is

$$U_t = E_t \left[\sum_{\tau=0}^{\infty} \rho_{t+\tau}^T q_{t+\tau} \right]$$

where the $n \times 1$ vector $\rho_{t+\tau} \equiv \left[\rho_{it+\tau}\right]$ is defined by a geometric random walk as

$$\rho_{it+\tau} = \rho_{it+\tau-1} F_{it+\tau}$$

where the F_{it} are non-negative *i.i.d.* demand shocks with $E\left[F_{it}\right] = \beta$ where β is the discount factor with $0 < \beta < 1$. We assume these demand shocks are independent of the supply shocks e_t. (If $Var\left[F_{it}\right] = 0$ then $\rho_{it+\tau} = \beta^\tau \rho_{it}$, the relative weights on each good are the same for all t, and this model makes the same predictions as the *cDGE* model.) Assuming ρ_t is observed at time t, we have from

$$\rho_{it+\tau} = F_{it+\tau} F_{it+\tau-1} F_{it+\tau-2} \times \cdots \times F_{it+1} \rho_{it}$$

that

$$E_t\left[\rho_{t+\tau}^T\right] = \beta^\tau \rho_t^T$$

and so

$$U_t = E_t \left[\sum_{\tau=0}^{\infty} \rho_{t+\tau}^T q_{t+\tau} \right] = \sum_{\tau=0}^{\infty} E_t\left[\rho_{t+\tau}^T\right] E_t\left[q_{t+\tau}\right] = \sum_{\tau=0}^{\infty} \beta^\tau \rho_t^T E_t\left[q_{t+\tau}\right] = E_t \left[\sum_{\tau=0}^{\infty} \beta^\tau \rho_t^T q_{t+\tau} \right].$$

This is the same as the *cDGE* model except that the time invariant ρ has been been replaced with ρ_t that follows a geometric random walk. Thus we have

$$q_{t+1} = b_t + A q_t + e_{t+1}$$

where the $n \times 1$ vector $b_t \equiv [b_{it}]$ is now given by

$$b_{it} = a_i + \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_{jt}^i\right) \text{ where } \gamma_{jt}^i = \frac{d_{it} a_{ij}}{\sum_{k=1}^{n} d_{kt} a_{kj}} \text{ and } d_t^T \equiv [d_{it}] = \frac{\rho_t^T (I - \beta A)^{-1}}{1 - \beta}.$$

The presence of the geometric random walks ρ_t in b_t will mean that the growth rate $v^T b_t$ will now evolve over time. This might then account for the structural breaks found in the empirical commodity price literature. As well we now have stochastic budget shares as

$$\frac{P_{it}Q_{it}}{Y_t} = \frac{d_{it}}{\sum_{i=1}^{n} d_{it}}$$

and so this might be able to account for non-constant budget shares when one has both price and quantity data.

The second approach is to add non-stationary supply shocks and modify the production functions of the *cDGE* model by replacing a_i with a non-stationary $\tilde{a}_{it}$ in (1) as

$$Q_{it} = \prod_{j=1}^{n} \left(Q_{jt-1}^{i}\right)^{a_{ij}} \exp\left(\tilde{a}_{it} + e_{it}\right) \text{ for all } i \in N$$

where the non-stationary $\tilde{a}_{it}$ experiences a break at time t_0 as

$$\tilde{a}_{it} = \begin{cases} a_i^1 & \text{for } t \leq t_0 \\ a_i^2 & \text{for } t > t_0 \end{cases} .$$

In this case we have

$$b_{it} = \tilde{a}_{it} + \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_j^i\right) = \begin{cases} a_i^1 + \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_j^i\right) & \text{for } t \leq t_0 \\ a_i^2 + \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_j^i\right) & \text{for } t > t_0 \end{cases}$$

and so there will be a change in the growth rate vb_t of the economy at $t = t_0$, thus accounting for the structural break observed in the data.

8 Conclusions

If one wants to do traditional time series econometrics with either price data or quantity data, then there is no need to consider anything more complicated than the *cDGE* model considered in this paper. Even if there are effects in the data that are not accounted for by the *TSE* model, we have shown that it is possible to amend the *cDGE* model to take into account these effects, and yet still have a closed-form solution.

There can be good scientific grounds for adopting models without closed-form solutions. Newton's law of gravity with more than two bodies does not have a closed-form solution, and it is able to predict effects that the two-body theory, which does have a closed-form solution, cannot. For those who believe in *DGE* models without closed-form solutions, the scientific challenge is to demonstrate the existence of trans-*TSE* effects that could not be accounted for by either the *cDGE* model, or a modified version of the *cDGE* model with a closed-form solution.

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10 Appendix

10.1 Derivation of the Input Weights

Define $\gamma_{jt}^i \equiv \frac{Q_{jt}^i}{Q_{jt}}$ as the share of Q_{jt} allocated for the production of good i at time $t+1$, chosen optimally to maximize U_t. Using $Q_{jt}^i \equiv \gamma_{jt}^i Q_{jt}$ we have

$$q_{t+1} = b_t + Aq_t + e_{t+1}$$

where the $n \times 1$ vector $b_{t+k} \equiv [b_{it+k}]$ is defined by

$$b_{it+k} \equiv a_i + \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_{jt+k}^i\right) \text{ for all } i \in N.$$

A competitive equilibrium will generate a Pareto optimal allocation, and so γ_{jt}^i will maximize utility U_t. We have

$$E_t\left[q_{t+k}\right] = \sum_{j=0}^{k-1} A^j b_{t+k-1-j} + A^k q_t + E_t\left[e_{t+k}\right] = A^k q_t + A^{k-1} b_t + \Gamma_t^k$$

where $\Gamma_t^k = 0$ for $k < 2$ while for $k \geq 2$

$$\Gamma_t^k \equiv \sum_{j=0}^{k-2} A^j b_{t+k-1-j} + E_t\left[e_{t+k}\right].$$

We therefore have

$$\begin{aligned} U_t &= \rho^T \sum_{k=0}^{\infty} \beta^k \left[A^k q_t + A^{k-1} b_t \ + \Gamma_t^k \right] = \rho^T \left(I - \beta A \right)^{-1} q_t + \beta \left(I - \beta A \right)^{-1} b_t + \rho \sum_{k=0}^{\infty} \beta^k \Gamma_t^k \\ &= \rho^T \left(I - \beta A \right)^{-1} q_t + d \times b_t + \rho \sum_{k=0}^{\infty} \beta^k \Gamma_t^k \end{aligned}$$

where

$$d = (1-\beta)\,\rho\,(I-\beta A)^{-1} = (1-\beta)\,\rho \sum_{k=0}^{\infty} \beta^k A^k.$$

The convergence of all infinite sums is assured by the fact the eigenvalues of A are bounded by one in absolute value and that $\beta < 1$. All elements of $(I-\beta A)^{-1}$ are non-negative and hence all elements of d are non-negative and sum to one or $d\iota_n = 1$.

To maximize U_t with respect to γ_{jt}^i we maximize $d \times b_t$ subject to $\sum_{i=1}^n \gamma_{jt}^i = 1$ which is equivalent to maximizing

$$\sum_{i=1}^{n} d_i \sum_{j=1}^{n} a_{ij} \ln\left(\gamma_{jt}^i\right) \text{ subject to } \sum_{i=1}^{n} \gamma_{jt}^i = 1 \text{ for all } j \in N.$$

The time invariant solution to this problem is

$$\gamma_{jt}^i = \frac{d_i a_{ij}}{\sum_{k=1}^n d_k a_{kj}} \text{ for all } i,j \in N.$$

At time t the representative household owns Q_{jt-1}. Let W_{jt} be the price of input j and suppose

$$P_{it} = d_i \frac{Y_t}{Q_{it}} \text{ where } Y_t \equiv \sum_{i=1}^{n} P_{it} Q_{it}.$$

Firms are perfectly competitive and make zero profits since there is constant returns to scale. Hence $Y_t = \sum_{j=1}^n W_{jt} Q_{jt-1}$. Profit maximization requires

$$P_{it} \frac{\partial Q_{it}}{\partial Q_{jt-1}^i} = W_{jt} \tag{23}$$

but

$$W_{jt} = P_{it} \frac{\partial Q_{it}}{\partial Q_{jt-1}^i} = P_{it} \frac{a_{ij} Q_{it}}{Q_{jt-1}^i} = \frac{Y_t d_i a_{ij}}{Q_{jt-1}^i} \tag{24}$$

or

$$Q_{jt-1}^i W_{jt} = Y_t d_i a_{ij}.$$

Applying $\sum_{i=1}^n$ to both sides then yields

$$W_{jt} = \frac{Y_t \sum_{i=1}^n d_i a_{ij}}{Q_{jt-1}}. \tag{25}$$

This in turn implies that

$$\frac{Q^i_{jt-1}}{Q_{jt-1}} = \frac{d_i a_{ij}}{\sum_{i=1}^n d_k a_{kj}} \equiv \gamma^i_j$$

and hence this set of markets supports this Pareto-optimal allocation of resources.

11 Seasonality

Complex eigenvalues where $|\lambda_k| = 1$ but $\lambda_k \neq 1$ cause endogenous seasonal dynamics, which we briefly describe here. In this case A (or one of the submatrices A^j_j for $j \geq 1$ in (16)) must be cyclic; that is by appropriately ordering the goods in N is must be possible to write A as

$$A = \begin{bmatrix} 0 & A^2_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A^s_{s-1} \\ A^1_s & 0 & \cdots & 0 \end{bmatrix}.$$

In this case $\lambda_k = e^{i\pi k/s}$ for $k = 0, \ldots, s-1$ (see Debreu and Herstein (1953)). If we then partition the elements of q_t, b, e_t in the same way as

$$\begin{aligned} q_t^T &= \begin{bmatrix} (q_t^1)^T & (q_t^2)^T & \cdots & (q_t^s)^T \end{bmatrix} \\ b^T &= \begin{bmatrix} (b^1)^T & (b^2)^T & \cdots & (b^s)^T \end{bmatrix} \\ e_t^T &= \begin{bmatrix} (e_t^1)^T & (e_t^2)^T & \cdots & (e_t^s)^T \end{bmatrix} \end{aligned}$$

then

$$\begin{aligned} q^i_t &= b^i + A^{i+1}_i q^{i+1}_{t-1} + e^i_t \text{ for } i = 1, 2, \ldots s-1 \\ q^s_t &= b^i + A^1_s q^1_{t-1} + e^i_t. \end{aligned}$$

Here q^i_t is the production of season i goods, which only have as inputs goods from the previous season.