General Equilibrium with Imperfect Competition

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September 6, 2013

Abstract

A computable general equilibrium model is constructed with imperfect competition and with heterogeneous firms, tastes, endowments, and technologies. The number of firms in each market is endogenous. The equilibrium is shown to be unique and easily computed. Necessary and sufficient conditions for efficiency are derived. Efficiency is possible, but requires that the rate of profit be the same for all industries.

1 Introduction

Positive results from the theory of imperfect competition are almost all based on partial equilibrium (for example Clarke and Davies (1982), Cowling Waterson (1976), and Mankiw and Whinston (1986)). With partial equilibrium non-trivial results can be derived using general functional forms, and these results in turn form the basis for much of the thinking behind competition policy. The most basic of these results is that imperfect competition is inefficient because firms restrict output. But does this conclusion hold up in a general equilibrium setting? Merely restricting output frees resources that can be used in other sectors of the economy. The increase in profits due to restricting output is paid as dividends that can be spent on goods in all sectors of the economy. It is hardly obvious from a general equilibrium perspective that imperfect competition is inefficient.

The general equilibrium literature on imperfect competition is mostly about negative results: the many technical problems that arise when general equilibrium and imperfect competition are combined. Gabszewocz and Vial (1972) show that the form of the demand curve (and hence the equilibrium solution) depends on the choice of numeraire. Firms pay profits to households in the form of dividends, which in turn affects the demand curve, the so-called Ford effect, and this must be accounted for when the firm makes its decisions. In a general equilibrium setting a firm must not only take into account the rivals within its own industry, but also all firms in all other industries. Imperfect competition, as noted by Arrow (1986), becomes an intractable mega-game where every agent in the economy has to take into account the effect of its actions on all other agents in the economy. From such intractable complexity it is difficult to even prove the existence of an equilibrium, let alone determine whether the equilibrium is efficient or not. (See Bonanno (1990) for a detailed survey.)

In this paper we derive positive results using general equilibrium. We construct a computable general equilibrium model with imperfect competition. We prove that the equilibrium exists, is unique, and we derive necessary and sufficient conditions for the equilibrium to be efficient. We allow unrestricted heterogeneity, within the assumed functional forms, in household endowments, equity ownership, and utility. We assume constant returns to scale, but otherwise allow unrestricted heterogeneity in the production function parameters across industries. Free entry is allowed, so that the number of firms in each industry is determined endogenously.

When this is done it turns out that the idea that imperfect competition is inefficient because firms restrict output is actually wrong. Given the right conditions imperfect competition can be efficient, and the paper derives the conditions under which this occurs. Even when imperfect competition is inefficient it is not because firms restrict output. Instead it is either because imperfect competition allows inefficient firms to operate, or because there are differential rates of profit across industries. Differential rates of profit create inefficiencies in a manner almost identical to the way that differential tax rates are inefficient in the optimal taxation literature. Efficiency requires a uniform rate of profit across industries, just as efficient taxation requires a uniform rate of taxation, as in Atkinson and Stiglitz (1976). This points to the possibility that insights in the optimal taxation literature could be used to construct an optimal competition policy literature.

2 The Model

Consider an economy with a finite set of households $\mathcal{H} = \{1, \ldots, H\}$ indexed by i, a finite set of goods $\mathcal{G} = \{1, \ldots, G\}$ indexed by j, and a finite set of factors of production $\mathcal{F} = \{1, \ldots, F\}$ indexed by l. The total amount of the good produced in industry $j \in \mathcal{G}$ is Q_j, with a finite set of potential firms $\mathcal{N}_j = \{1, \ldots, N_j\}$ indexed by m, and where $N_j > 1$. Income of household $i \in \mathcal{H}$ is y_i, and the total amount of factor $l \in \mathcal{F}$ is E_l, which we normalize so that $E_l \equiv 1$ for all $l \in \mathcal{F}$. The price of good $j \in \mathcal{G}$ is p_j, and the price of factor $l \in \mathcal{F}$ is ω_l. The value of production in industry j is $v_j \equiv p_j Q_j$. Define the $1 \times G$ vector $v \equiv [v_j]$, the $1 \times F$ vector $\omega \equiv [\omega_l]$, and let $\iota_n = [1]$ be an $n \times 1$ vector of ones. We normalize prices so that the total value of all goods produced is one or

$$v\iota_G = 1. \tag{1}$$

Let Q_{ij} be household $i's$ demand for good j with Cobb-Douglas utility function

$$U_i = \sum_{j=1}^{G} \alpha_{ij} \ln(Q_{ij}) \text{ with } \alpha_{ij} > 0 \text{ and } \sum_{j=1}^{G} \alpha_{ij} = 1. \tag{2}$$

The $H \times G$ matrix $A \equiv [\alpha_{ij}]$ is a stochastic matrix, that is the elements are positive with rows that sum to one as

$$A\iota_G = \iota_H. \tag{3}$$

Household i has an endowment $E_{li} > 0$ of factor l. From $\sum_{i=1}^{H} E_{li} = E_l = 1$ it follows that the $F \times H$ matrix $E \equiv [E_{li}]$ is a stochastic matrix or

$$E\iota_H = \iota_F. \tag{4}$$

Income y_i of household i consists of factor income $\sum_{l=1}^{F} \omega_l E_{li}$ and dividend income δ_i. Each household is endowed with an exogenous share $\mu_{ji}(m) \geq 0$ of the profits of firm $m \in \mathcal{N}_j$ with $\sum_{i=1}^{H} \mu_{ji}(m) = 1$. If $\pi_j(m)$ is the profits of firm m in industry j, then

$$\delta_i = \sum_{j=1}^{G} \sum_{m=1}^{N_j} \mu_{ji}(m) \pi_j(m).$$

Let $y \equiv [y_i]$ and $\delta = [\delta_i]$ be the $1 \times H$ vectors of household income and household dividend income. We then have

$$y = \omega E + \delta. \tag{5}$$

Households are price takers so that utility maximization gives $Q_{ij} = \frac{y_i \alpha_{ij}}{p_j}$ or $p_j Q_{ij} = y_i \alpha_{ij}$. Summing over i and using $\sum_{i=1}^{H} Q_{ij} = Q_j$ yields

$$v = yA. \tag{6}$$

From $A\iota_G = \iota_H$ and $v\iota_G = 1$ it follows that $y\iota_H = 1$.

2.1 Firms

Production of firm $m \in \mathcal{N}_j$ in industry $j \in \mathcal{G}$ is $Q_j(m)$ with market share $s_j(m) \equiv \frac{Q_j(m)}{Q_j}$. Firms in each industry differ according to an efficiency parameter $\Gamma_j(m) > 0$ with a constant returns to scale Cobb-Douglas production function

$$Q_j(m) = \frac{1}{\Gamma_j(m)} \prod_{l=1}^{F} K_{jl}(m)^{\beta_{jl}} \text{ with } \beta_{jl} > 0 \text{ and } \sum_{l=1}^{F} \beta_{jl} = 1 \text{ for } m \in \mathcal{N}_j$$

where $K_{jl}(m)$ is the amount of factor l used by firm m in industry j. Firms are assumed to be competitive in factor markets so that firm $m's$ cost function is

$$C_j(m) = \Gamma_j(m) Q_j(m) \prod_{l=1}^{F} \left(\frac{\omega_l}{\beta_{jl}}\right)^{\beta_{jl}}. \tag{7}$$

Define the $G \times F$ matrix $B \equiv [\beta_{jl}]$. Since there are constant returns to scale the matrix B is a stochastic matrix as

$$B\iota_F = \iota_G. \tag{8}$$

We assume that firms are ordered from most to least efficient as

$$0 < \Gamma_j(1) \leq \Gamma_j(2) \leq \cdots \leq \Gamma_j(N_j).$$

Let N_j^+ be the number of firms in industry $j \in \mathcal{G}$ that produce positive output (i.e., $Q_j(m) > 0$), and let $\mathcal{N}_j^+ \equiv \{m | Q_j(m) > 0\}$ be the set of these firms. There is free entry so that N_j^+ is determined endogenously; that is each of the N_j firms decides whether to enter with $Q_j(m) > 0$ or not to enter with $Q_j(m) = 0$.

We will see that N_j^+ is determined by the relative efficiencies of the marginal and average firms. For any given n and

$$\Gamma_j(1) \leq \Gamma_j(2) \leq \cdots \leq \Gamma_j(n)$$

we define $\Gamma_j(n)$ as the efficiency of the marginal firm and $\bar{\Gamma}_j(n)$ as the efficiency of the average firm. It will be convenient to use $n-1$ instead of the conventional n in the denominator of $\bar{\Gamma}_j(n)$ as

$$\bar{\Gamma}_j(n) \equiv \frac{1}{n-1} \sum_{m=1}^{n} \Gamma_j(m).$$

We define $\Gamma_j^* \equiv \Gamma_j(1)$ as the efficiency parameter of the most efficient firms, $\mathcal{N}_j^* = \{m | \Gamma_j(m) = \Gamma_j^*\}$ as the set of efficient firms, N_j^* as the number of efficient firms, and $s_j^* \equiv \sum_{m \in \mathcal{N}_j^*} s_j(m)$ as the market share of the efficient firms.

Since $v_j \equiv p_j Q_j$, the industry inverse demand curve is

$$p_j = \frac{v_j}{Q_j} = \frac{v_j}{\sum_{m=1}^{N_j} Q_j(m)}. \tag{9}$$

We assume that each firm in industry j facing (9) adopts Cournot behavior because of the presence of their rivals' output $Q_j = \sum_{m=1}^{N_j} Q_j(m)$ in the denominator. Since $\sum_{j=1}^{G} v_j = 1$, the numerator v_j in (9) is industry $j's$ share of total income $Y = 1$. Rather than being price-takers as in the competitive case, we assume that firms are industry-share takers; that is they treat v_j in (9) as being independent of their actions.

Industry-Share taking Assumption: *For all $j \in \mathcal{G}$, firms treat v_j in (9) as being independent of their actions.*

This assumption is a natural one to make in a Cobb-Douglas setting. It allows non-competitive behavior without creating the intractable general equilibrium problems discussed in the introduction. Effectively it provides a cut (as does the partial equilibrium assumption) between firms' actions within an industry and what happens outside that industry, so that a firm in industry j need only worry about its rivals in industry j, and not firms in other industries.

Clearly one needs to assume some cut somewhere if one wants to obtain positive results. Indeed not to assume any such cut is highly unrealistic since effectively each firm and household would have to take into account the effect of its actions on all other firms and all other households in the entire world. One can imagine leather manufacturers in Argentina competing with each other, but they surely do not compete with milk producers in India.

Since $p_j \equiv \frac{v_j}{Q_j}$ with v_j fixed, profits for firm m are

$$\pi_j(m) = \frac{v_j}{\sum_{m=1}^{N_j} Q_j(m)} Q_j(m) - \Gamma_j(m) Q_j(m) \prod_{l=1}^{F} \left(\frac{\omega_l}{\beta_{jl}}\right)^{\beta_{jl}} \quad \text{for } m \in \mathcal{N}_j. \tag{10}$$

We have

$$\frac{\partial \pi_j(m)}{\partial Q_j(m)} = \frac{v_j}{Q_j}\left(1 - \frac{Q_j(m)}{Q_j}\right) - \Gamma_j(m) \prod_{l=1}^{F} \left(\frac{\omega_l}{\beta_{jl}}\right)^{\beta_{jl}}.$$

It is easily verified that $\frac{\partial^2 \pi_j(m)}{\partial Q_j(m)^2} < 0$, so $\pi_j(m)$ is a strictly concave function of $Q_j(m)$, and hence any solution to the first-order condition with $Q_j(m) > 0$ is a global maximum. From the first-order conditions we have

$$\frac{v_j}{Q_j}\left(1 - \frac{Q_j(m)}{Q_j}\right) = \Gamma_j(m) \prod_{l=1}^{F} \left(\frac{\omega_l}{\beta_{jl}}\right)^{\beta_{jl}} \quad \text{for } m \in \mathcal{N}_j^+. \tag{11}$$

Summing over $m \in \mathcal{N}_j^+$ yields

$$\prod_{l=1}^{F} \left(\frac{\omega_l}{\beta_{jl}}\right)^{\beta_{jl}} = \frac{1}{\bar{\Gamma}_j\left(N_j^+\right)} \frac{v_j}{Q_j}. \tag{12}$$

Putting (12) in (10) and (11) yields profits

$$\pi_j(m) = v_j \frac{Q_j(m)}{Q_j} \left(1 - \frac{\Gamma_j(m)}{\bar{\Gamma}_j(N_j^+)}\right) \text{ for } m \in \mathcal{N}_j^+ \tag{13}$$

and market shares

$$s_j(m) = 1 - \frac{\Gamma_j(m)}{\bar{\Gamma}_j(N_j^+)} \text{ for } m \in \mathcal{N}_j^+. \tag{14}$$

From (14) it follows that the more efficient a firm is, the larger is its market share.

For any solution with a given N_j^+ to be a Nash equilibrium it must be that $s_j(m) > 0$ for all $m \in \mathcal{N}_j^+$, or equivalently $\Gamma_j(N_j^+) < \bar{\Gamma}_j(N_j^+)$. As well no firm with $m \notin \mathcal{N}_j^+$ should have an incentive to enter. From (13) this will be satisfied if

$$\Gamma_j(m) \geq \bar{\Gamma}_j(N_j^+) \text{ for } m \notin \mathcal{N}_j^+$$

in which case $Q_j(m) = 0$ for $m \notin \mathcal{N}_j^+$ maximizes profits.

In the Appendix we prove two lemmas given below. The two lemmas show that $\Gamma_j(n)$ and $\bar{\Gamma}_j(n)$ have a conventional marginal-average relationship as shown in the Figure 1 below.

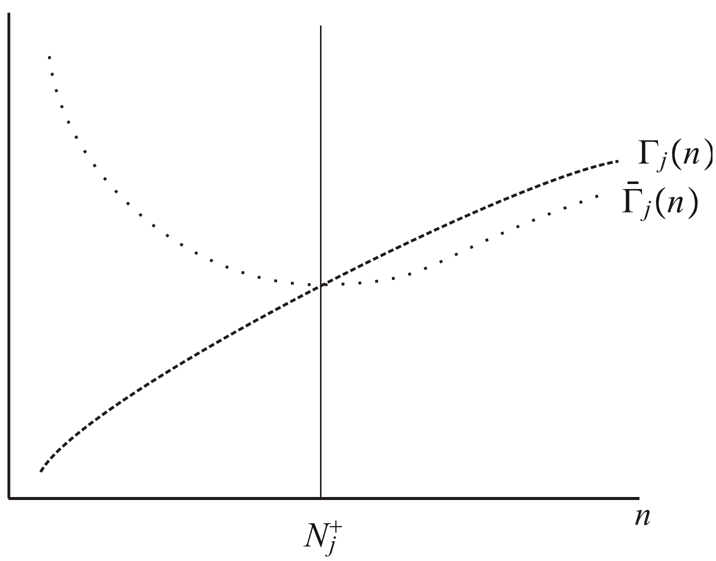

Figure 1

Lemma 1 *If $\Gamma_j(n) < \bar{\Gamma}_j(n)$ for $n \geq 2$, then for all $2 \leq m \leq n$*

$$\Gamma_j(m) < \bar{\Gamma}_j(m) \text{ and } \bar{\Gamma}_j(m) < \bar{\Gamma}_j(m-1).$$

Lemma 2 *If $\Gamma_j(n) \geq \bar{\Gamma}_j(n)$ then for all $m \geq n$*

$$\Gamma_j(m) \geq \bar{\Gamma}_j(m) \ \text{and} \ \bar{\Gamma}_j(m+1) \geq \bar{\Gamma}_j(m).$$

Basically if $\Gamma_j(n) < \bar{\Gamma}_j(n)$ then $\bar{\Gamma}_j(m)$ is decreasing for $m \leq n$ and the marginal firm with $\Gamma_j(n)$, finding that it is more efficient than the average firm with $\bar{\Gamma}_j(n)$, enters and causes $\bar{\Gamma}_j(n)$ to fall. Entry continues until the marginal firm does not find it profitable to enter, with N_j^+ determined by the intersection of $\Gamma_j(n)$ and $\bar{\Gamma}_j(n)$ as in Figure 1. This is summarized in the following result proven in the Appendix.

Theorem 3 *A Nash equilibrium exists with $N_j^+ \geq 2$, it is unique, and is the maximum n that satisfies*

$$\Gamma_j(n) < \bar{\Gamma}_j(n).$$

From (13) and (14) costs and profits for firm m are

$$C_j(m) = v_j s_j(m)(1 - s_j(m)) \ \text{and} \ \pi_j(m) = v_j s_j(m)^2. \tag{15}$$

Total profits in industry j are

$$\pi_j \equiv \sum_{m \in \mathcal{N}_j^+} \pi_j(m) = v_j \sum_{m \in \mathcal{N}_j^+} s_j(m)^2 = v_j \tilde{\pi}_j$$

where

$$\tilde{\pi}_j \equiv \sum_{m \in \mathcal{N}_j^+} s_j(m)^2 = \frac{1}{N_j^+} + \sum_{m \in \mathcal{N}_j^+} \left(s_j(m) - \frac{1}{N_j^+} \right)^2 \tag{16}$$

is the rate of profit on sales, as well as the Herfindahl index of industry concentration. It follows that the rate of profit is bounded as $\frac{1}{N_j^+} \leq \tilde{\pi}_j < 1$ with $\tilde{\pi}_j$ attaining its lower bound $\tilde{\pi}_j = \frac{1}{N_j^+}$ if and only if $s_j^* = 1$ (all producing firms are efficient).

Using $C_j(m) = v_j s_j(m)(1 - s_j(m))$ and Shephard's lemma, the amount of factor l used by firm $m \in \mathcal{N}_j^+$ is

$$K_{jl}(m) = \frac{\partial C_j(m)}{\partial \omega_l} = \frac{v_j (1 - s_j(m)) s_j(m) \beta_{jl}}{\omega_l}. \tag{17}$$

Summing over $m \in \mathcal{N}_j^+$ yields the amount of factor l used by industry j as

$$K_{jl} = \frac{v_j (1 - \tilde{\pi}_j) \beta_{jl}}{\omega_l}. \tag{18}$$

Summing $\omega_l K_{jl}$ over $j \in \mathcal{G}$ and using $E_l = 1$ yields

$$\omega = v (I - \tilde{\pi}_d) B \tag{19}$$

where $\tilde{\pi}_d$ is a $G \times G$ diagonal matrix with $\tilde{\pi}_d \equiv diag[\tilde{\pi}_j]$.

Using (17) we have $Q_j(m) = s_j(m) Q_j$ where output of industry j is

$$Q_j = \frac{1}{\bar{\Gamma}_j\left(N_j^+\right)(1-\tilde{\pi}_j)} \prod_{l=1}^{F} K_{jl}^{\beta_{jl}} = \frac{1}{\Gamma_j} \prod_{l=1}^{F} K_{jl}^{\beta_{jl}} \tag{20}$$

and

$$\Gamma_j \equiv \bar{\Gamma}_j\left(N_j^+\right)(1-\tilde{\pi}_j) = \bar{\Gamma}_j\left(N_j^+\right) \sum_{m \in \mathcal{N}_j^+} s_j(m)(1-s_j(m)) = \sum_{m \in \mathcal{N}_j^+} s_j(m) \Gamma_j(m). \tag{21}$$

One of the inefficiencies of a non-competitive market structure is that it allows inefficient firms to operate. Here Γ_j is a measure of this technological inefficiency with $\Gamma_j \geq \Gamma_j^*$ and equality if and only if $s_j^* = 1$; that is if all firms in industry j use the most efficient technology.

2.2 Dividend Income

Household $i's$ share of the profits from industry j are

$$\mu_{ji} \equiv \frac{\sum_{m \in \mathcal{N}_j^+} s_j(m)^2 \mu_{ji}(m)}{\tilde{\pi}_j} \text{ with } \sum_{i=1}^{H} \mu_{ji} = 1.$$

Define the $G \times H$ matrix $M \equiv \left[\mu_{ji}\right]$, which is a stochastic matrix as

$$M\iota_H = \iota_G.$$

Because M is a function only of $\mu_{ji}(m), \Gamma_j(m)$ for all $i \in \mathcal{H}, m \in \mathcal{N}_j, j \in \mathcal{G}$, it can be calculated independently of the equilibrium v, ω, y. Dividend income for household i then is

$$\delta_i = \sum_{j=1}^{G} \sum_{m \in \mathcal{N}_j^+} v_j \mu_{ji}(m) s_j(m)^2 = \sum_{j=1}^{G} v_j \tilde{\pi}_j \mu_{ji}$$

and so from (5) we have

$$y = \omega E + v\tilde{\pi}_d M. \tag{22}$$

2.3 The General Equilibrium Solution

We now show that the equilibrium exists and is unique. The basic result is the following.

Theorem 4 *v is the unique strictly positive solution to $v = vD$ and $v\iota_G = 1$ where the $G \times G$ stochastic matrix D is given by*

$$D \equiv (I - \tilde{\pi}_d) BEA + \tilde{\pi}_d MA.$$

Proof. From (6), (19), (22) we have

$$v = yA, \ \omega = v\left(I - \tilde{\pi}_d\right)B, \text{ and } y = \omega E + v\tilde{\pi}_d M. \tag{23}$$

Thus

$$v = yA = \omega EA + v\tilde{\pi}_d MA = v\left(I - \tilde{\pi}_d\right)BEA + v\tilde{\pi}_d MA = v\left(\left(I - \tilde{\pi}_d\right)BEA + \tilde{\pi}_d MA\right).$$

As $0 \leq \tilde{\pi}_j < 1$ the elements of B, E, A are strictly positive. The elements of M are non-negative. It follows that the elements of D are strictly positive. Furthermore D is a stochastic matrix as

$$\begin{aligned} D\iota_G &= \left(I - \tilde{\pi}_d\right)BEA\iota_G + \tilde{\pi}_d MA\iota_G = \left(I - \tilde{\pi}_d\right)BE\iota_H + \tilde{\pi}_d M\iota_H \\ &= \left(I - \tilde{\pi}_d\right)B\iota_F + \tilde{\pi}_d\iota_G = \left(I - \tilde{\pi}_d\right)\iota_G + \tilde{\pi}_d\iota_G = \iota_G. \end{aligned}$$

By the Perron-Frobenius theorem (see Grimmett and Stirzaker (1982)) there exists a unique strictly positive vector v that satisfies $v\iota_G = 1$, and solves $v = vD$. ■

From v we can then solve for ω and y using (23). The vectors ω and y are also both strictly positive with $y\iota_H = 1$ and

$$\omega\iota_F = 1 - v\tilde{\pi}_d\iota_G = 1 - \tilde{\pi}$$

where the average rate of profit in the economy is

$$\tilde{\pi} = \sum_{j=1}^{G} v_j \tilde{\pi}_j. \tag{24}$$

Define $q_{ij} \equiv \frac{Q_{ij}}{Q_j}$ as the share of good j going to household i. The equilibrium allocation, denoted by $\{q_{ij}, K_{jl}, \Gamma_j\}$, is given by

$$q_{ij} = \frac{y_i \alpha_{ij}}{v_j}, K_{jl} = \frac{v_j\left(1 - \tilde{\pi}_j\right)\beta_{jl}}{\omega_l}, \Gamma_j = \sum_{m \in \mathcal{N}_j^+} s_j(m)\,\Gamma_j(m) \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}, l \in \mathcal{F}. \tag{25}$$

In the allocation $\{q_{ij}, K_{jl}, \Gamma_j\}$ the q_{ij} determine whether a given level of production Q_j is efficiently allocated amongst the H households; the K_{jl} determine whether the given factor endowments are efficiently allocated amongst the G industries; the Γ_j determine (relative to Γ_j^*) the technological efficiency of each industry.

Solving $v = vD$ is straightforward. One method is the recursion

$$v^t = v^{t-1}D$$

with any non-negative initial v^0 satisfying $v^0\iota_G = 1$. Since all eigenvalues of D except for $\lambda_1 = 1$ satisfy $|\lambda_j| < 1$, it follows that $v^t \to v$ exponentially. An alternative method that generates an exact solution is to solve the linear system of equations $v = vD$ for v. To this end partition D and v as

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

where D_{12} and D_{21} are column and row vectors with $G-1$ elements, and D_{22} and v_2 are scalars. Since the rows of D sum to 1, and since D_{12} has strictly positive elements, the matrix $(I - D_{11})^{-1} = \sum_{k=0}^{\infty} D_{11}^k$ exists and has strictly positive elements. The solution is then

$$v_1 = \frac{D_{21}\left(I - D_{11}\right)^{-1}}{1 + D_{21}\left(I - D_{11}\right)^{-1} \iota_{G-1}}, v_2 = \frac{1}{1 + D_{21}\left(I - D_{11}\right)^{-1} \iota_{G-1}}.$$

We note here that we have assumed that all elements of A, B, E are strictly positive. It is possible to allow weak inequalities and instead assume that BEA is indecomposable, which means that the economy cannot be split into non-interacting sub-economies. This is addressed in a competitive version of this model in Sampson (2013).

3 Efficiency

We now seek the conditions under which imperfectly competition allocation $\{q_{ij}, K_{jl}, \Gamma_j\}$ in (25) is efficient. Consider a benevolent central planner that maximizes social welfare U, the weighted sum of individual utilities as

$$U = \sum_{i=1}^{H} y_i^* U_i$$

where the $1 \times H$ vector of strictly positive utility weights $y^* \equiv [y_i^*]$ is chosen to satisfy the normalization $y^* \iota_H = 1$.

Let K_{jl}^* be the optimal amount of factor l allocated to industry j. The central planner uses the most efficient production functions available, so that if Q_j^* is the optimal production of good j then

$$Q_j^* = \frac{1}{\Gamma_j^*} \prod_{l=1}^{F} \left(K_{jl}^*\right)^{\beta_{jl}}.$$

Define Q_{ij}^* as the optimal amount of good j given to household i, and define $q_{ij}^* = \frac{Q_{ij}^*}{Q_j^*}$ as the share of good j. We have

$$U = \sum_{j=1}^{G} y_i^* \sum_{i=1}^{H} \alpha_{ij} \ln\left(q_{ij}^* Q_j^*\right) = \sum_{j=1}^{G} y_i^* \sum_{i=1}^{H} \alpha_{ij} \ln\left(q_{ij}^*\right) + \sum_{j=1}^{G} v_j^* \left(\sum_{l=1}^{F} \beta_{jl} \ln\left(K_{jl}^*\right) - \ln\left(\Gamma_j^*\right) \right).$$

The efficient allocation $\left\{q_{ij}^*, K_{jl}^*, \Gamma_j^*\right\}$ then solves

$$\max_{q_{ij}^*, K_{jl}^*} U \textit{ subject to } \sum_{i=1}^{H} q_{ij}^* = 1 \textit{ and } \sum_{j=1}^{G} K_{jl}^* = 1. \tag{26}$$

The solution to (26) is the allocation $\left\{q_{ij}^*, K_{jl}^*, \Gamma_j^*\right\}$ where

$$q_{ij}^* = \frac{y_i^* \alpha_{ij}}{v_j^*}, K_{jl}^* = \frac{v_j^* \beta_{jl}}{\omega_l^*} \tag{27}$$

and where the $1 \times G$ vector $v^* \equiv \left[v_j^*\right]$ and $1 \times F$ vector $\omega^* = [\omega_l^*]$ can be computed from the given y^* as

$$v^* \equiv y^* A \text{ and } \omega^* \equiv v^* B = y^* AB.$$

We now seek the conditions under which $\{q_{ij}, K_{jl}, \Gamma_j\}$ and $\left\{q_{ij}^*, K_{jl}^*, \Gamma_j^*\right\}$ are the same for some strictly positive y^* satisfying $y^* \iota_H = 1$. We first show that if $\{q_{ij}, K_{jl}, \Gamma_j\}$ is efficient it can only be relative to the observed distribution of income; that is for $y^* = y$, and hence $v = v^*$. The proof is as follows: Suppose for all $i \in \mathcal{H}, j \in \mathcal{G}$ that

$$\frac{y_i \alpha_{ij}}{v_j} \equiv q_{ij} = q_{ij}^* \equiv \frac{y_i^* \alpha_{ij}}{v_j^*}$$

for some y^*. Since $\alpha_{ij} > 0$ it follows that $\frac{y_i}{v_j} = \frac{y_i^*}{v_j^*}$. Summing over i and using $\sum_{i=1}^{H} y_i = \sum_{i=1}^{H} y_i^* = 1$ yields $v_j = v_j^*$ for all $j \in \mathcal{G}$, which in turn implies that $y_i = y_i^*$ for all $i \in \mathcal{H}$.

We now show that for the imperfectly competitive equilibrium $\{q_{ij}, K_{jl}, \Gamma_j\}$ to be efficient it must be that the rate of profit is the same for all industries.

Theorem 5 *If the allocation* $\{q_{ij}, K_{jl}, \Gamma_j\}$ *is efficient then* $\tilde{\pi}_j = \tilde{\pi}$ *for all* $j \in \mathcal{G}$.

Proof. Given $v^* = v$ we have

$$\frac{v_j (1 - \tilde{\pi}_j) \beta_{jl}}{\omega_l} = K_{jl} = K_{jl}^* = \frac{v_j \beta_{jl}}{\omega_l^*} \text{ for all } j \in \mathcal{G}, l \in \mathcal{F}.$$

Since $\beta_{jl} > 0$ it follows that $(1 - \tilde{\pi}_j) \omega_l^* = \omega_l$. Summing over l and using $\sum_{l=1}^{F} \omega_l^* = 1$ and $\sum_{l=1}^{F} \omega_l = 1 - \tilde{\pi}$ we have $\tilde{\pi}_j = \tilde{\pi}$ for all $j \in \mathcal{G}$. ■

Finally for the imperfectly competitive equilibrium $\{q_{ij}, K_{jl}, \Gamma_j\}$ to be efficient it must be that all firms with positive output are efficient, or from (25) that

$$\Gamma_j = \sum_{m=1}^{N_j^+} s_j(m) \Gamma_j(m) = \Gamma_j^*.$$

This can only be true if $\Gamma_j(m) = \Gamma_j^*$ and $s_j(m) = \frac{1}{N_j^*}$ for all $m \in N_j^+ = N_j^*$. From (16) the rate of profit is then $\tilde{\pi}_j = \frac{1}{N_j^*}$ for all $j \in \mathcal{G}$. But we have already seen that the rates of profits must be equalized and so it must be that for some $N^* > 1$ that $N_j^* = N^*$ for all $j \in \mathcal{G}$.

That is if the allocation $\{q_{ij}, K_{jl}, \Gamma_j\}$ is to be efficient it must be that each industry in the economy has an identical number of firms N^*, that each of these firms uses the most efficient technology with $\Gamma_j = \Gamma_j^*$, and consequently that the rate of profit is the same in all industries with $\tilde{\pi}_j = \frac{1}{N^*}$ for all $j \in \mathcal{G}$.

In addition it must that no inefficient firm has an incentive to produce. This will be satisfied if for all $j \in \mathcal{G}$

$$\Gamma_j(N^*+1) > \bar{\Gamma}_j(N^*) = \frac{N^*}{N^*-1}\Gamma_j^* \text{ or } \frac{\Gamma_j(N^*+1)}{\Gamma_j^*} > \frac{N^*}{N^*-1}.$$

This leads us to the following necessary and sufficient conditions for efficiency of the imperfectly competitive equilibrium.

Theorem 6 *The allocation $\{q_{ij}, K_{jl}, \Gamma_j\}$ in (25) is efficient if and only if for all $j \in \mathcal{G}$ there is a $N^* > 1$ such that 1) $N_j^* = N^*$ (and so $\tilde{\pi}_j = \frac{1}{N^*}$) and 2) $\frac{\Gamma_j(N^*+1)}{\Gamma_j^*} > \frac{N^*}{N^*-1}$.*

4 Optimal Competition Policy

It follows from Theorem 6 that imperfect competition can be efficient. Any market structure from duopoly $N^* = 2$ to perfect competition $N^* = \infty$ is consistent with efficiency as long as 1) there are an identical number of firms in each industry and hence the same rate of profit across industries, and 2) all firms that are producing use the most efficient technology Γ_j^*. If these conditions do not hold then the equilibrium will be inefficient and there is room for competition policy to improve welfare. But the reason the equilibrium would be inefficient is not because firms restrict output. Rather the inefficiency is due to imperfect competition allowing inefficient firms with $\Gamma_j(m) > \Gamma_j^*$ to operate and make positive profits, or to imperfect competition allowing differential rates of profit $\tilde{\pi}_j \neq \tilde{\pi}$ to distort factor allocation across industries. An industry with a below average rate of profit $\tilde{\pi}_j < \tilde{\pi}$ restricts output *too little* for efficiency.

The rate of profit $\tilde{\pi}_j$ plays a similar role to a tax rate τ_j with profits are paid to shareholders instead of the government. The requirement for a uniform rate of profit is similar to the requirement for uniform commodity taxation in the optimal taxation literature, as in Atkinson and Stiglitz (1976). In a competitive version of this model with taxation (see Sampson (2007, 2)) it is shown that a uniform tax rate $\tau_j = \tau$ is required for efficiency, and for the same reason we obtain the uniform rate of profit result: to insure factors of production are optimally allocated across industries.

Both optimal taxation and optimal competition policy are fundamentally general equilibrium problems. Just as it is not optimal to adjust rates of taxation on an industry-by-industry basis, so too it is not optimal to adjust market structure on an industry-by-industry basis. To illustrate suppose all industries except industry 1 satisfy Theorem 6 as

$$N_j^+ = N_j^* = N^* \text{ for } 2 \leq j \leq G \text{ and } N_1^+ = N_1^* > N^*.$$

In this case industry 1 has too many firms, or equivalently too low a rate of profit, as

$$\tilde{\pi}_1 = \frac{1}{N_1^*} < \tilde{\pi} = \frac{1}{N^*}.$$

Industry 1 is not restricting output enough. Competition policy that made industry 1 more competitive by increasing N_1^* or reducing $\tilde{\pi}_1$ would take the economy farther away from the efficiency conditions in Theorem 6.

Optimal competition policy in a general equilibrium setting is more difficult than suggested by partial equilibrium. In our model optimal competition policy would somehow have to: 1) induce inefficient firms to exit and 2) insure that each industry has the same number of efficient firms and hence the same rate of profit. If competition policy succeeded in doing this, the resulting equilibrium would be efficient. But it will also have effects on income y, and these income effects mean that there will generally be losers relative to the inefficient status quo equilibrium. It will then be in the interest of these losers to block implementation of this policy. So optimal competition policy may not be politically viable unless in addition to the requirements of Theorem 6 it is able to achieve a Pareto dominating allocation where there are no losers.

It is possible remove the complicating income effects of competition policy by imposing two restrictions on the model. The first is to assume that households invest by industry and not by firm so that

$$\mu_{ji}(m) = \mu_{ji} \text{ for all } i \in \mathcal{H}, j \in \mathcal{G}, m \in \mathcal{N}_j.$$

This means that the matrix $M = [\mu_{ji}]$ is independent of the $\Gamma_j(m)' s$, and so is invariant to any changes in market structure caused by competition policy. The second restriction is stronger, and is to assume that M takes the particular form

$$M = BE.$$

This assumption insures that any change in dividend income δ_i caused by competition policy is exactly compensated for by a change in factor income $\sum_{l=1}^{F} \omega_l E_{li}$. This follows since if $M = BE$ then

$$D = (I - \tilde{\pi}_d) BEA + \tilde{\pi}_d MA = (I - \tilde{\pi}_d) BEA + \tilde{\pi}_d BEA = BEA$$

and so v in $v = vD$ is invariant to competition policy, and hence so is

$$y = v(I - \tilde{\pi}_d) BE + v\tilde{\pi}_d M = v(I - \tilde{\pi}_d) BE + v\tilde{\pi}_d BE = vBE.$$

For such an economy optimal competition policy is much simpler, it need only worry about insuring that conditions of Theorem 6 are satisfied.

5 Conclusions

We have developed a computable general equilibrium model with imperfect competition, and with it we have derived positive results. An equilibrium exists, it is unique, and it can be easily computed. Contrary to the intuition from partial equilibrium, the resulting equilibrium can be efficient. When it is inefficient, it is not because firms restrict output, but rather because inefficient firms are allowed to operate, or because there are differential rates of profit across sectors. The close link between this result and the efficiency of uniform taxation in the optimal taxation literature suggests the possibility of a similar theory of optimal competition policy.

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A Appendix

A.1 Proof of Lemma 1

We seek to prove that if $\Gamma_j(n) < \bar{\Gamma}_j(n)$ for $n \geq 2$, then for all $2 \leq m \leq n$

$$\Gamma_j(m) < \bar{\Gamma}_j(m) \text{ and } \bar{\Gamma}_j(m) < \bar{\Gamma}_j(m-1).$$

The proof is as follows: For $n = 2$ the result holds as

$$\Gamma_j(2) < \bar{\Gamma}_j(2) = \Gamma_j(1) + \Gamma_j(2) \text{ and } \bar{\Gamma}_j(2) = \Gamma_j(1) + \Gamma_j(2) < \bar{\Gamma}_j(1) = \infty.$$

Suppose $\Gamma_j(n) < \bar{\Gamma}_j(n)$ for $n > 2$. Then

$$\bar{\Gamma}_j(n) = \frac{n-2}{n-1}\bar{\Gamma}_j(n-1) + \frac{\Gamma_j(n)}{n-1} < \frac{n-2}{n-1}\bar{\Gamma}_j(n-1) + \frac{\bar{\Gamma}_j(n)}{n-1}$$

and so

$$\left(1 - \frac{1}{n-1}\right)\bar{\Gamma}_j(n) = \frac{n-2}{n-1}\bar{\Gamma}_j(n) < \frac{n-2}{n-1}\bar{\Gamma}_j(n-1).$$

Since $\frac{n-2}{n-1} > 0$ for $n > 2$, the result $\bar{\Gamma}_j(n) < \bar{\Gamma}_j(n-1)$ follows. As well $\Gamma_j(n-1) < \bar{\Gamma}_j(n-1)$ since

$$\Gamma_j(n-1) \leq \Gamma_j(n) < \bar{\Gamma}_j(n) < \bar{\Gamma}_j(n-1).$$

The general result then follows by induction.

A.2 Proof of Lemma 2

Here we prove that if $\Gamma_j(n) \geq \bar{\Gamma}_j(n)$ then for all $m \geq n$

$$\Gamma_j(m) \geq \bar{\Gamma}_j(m) \text{ and } \bar{\Gamma}_j(m+1) \geq \bar{\Gamma}_j(m).$$

The proof is as follows: Suppose that $\Gamma_j(n) \geq \bar{\Gamma}_j(n)$ for $n > 2$. Since $\Gamma_j(n) \leq \Gamma_j(n+1)$ we have

$$\begin{aligned} \bar{\Gamma}_j(n+1) &= \frac{n-1}{n}\bar{\Gamma}_j(n) + \frac{\Gamma_j(n+1)}{n} \leq \frac{n-1}{n}\Gamma_j(n) + \frac{\Gamma_j(n+1)}{n} \\ &\leq \frac{n-1}{n}\Gamma_j(n+1) + \frac{\Gamma_j(n+1)}{n} = \Gamma_j(n+1) \end{aligned}$$

and so $\Gamma_j(n+1) \geq \bar{\Gamma}_j(n+1)$ follows. From this result we then have

$$\bar{\Gamma}_j(n+1) = \frac{n-1}{n}\bar{\Gamma}_j(n) + \frac{\Gamma_j(n+1)}{n} \geq \frac{n-1}{n}\bar{\Gamma}_j(n) + \frac{\bar{\Gamma}_j(n+1)}{n}$$

so that

$$\left(1 - \frac{1}{n}\right)\bar{\Gamma}_j(n+1) = \frac{n-1}{n}\bar{\Gamma}_j(n+1) \geq \frac{n-1}{n}\bar{\Gamma}_j(n)$$

and so $\bar{\Gamma}_j(n+1) \geq \bar{\Gamma}_j(n)$ follows. The result then follows by induction.

A.3 Proof of Theorem 3

Define $\hat{n}_j$ as the maximum n that satisfies $\Gamma_j(n) < \bar{\Gamma}_j(n)$, which exists since $N_j < \infty$. We have $\hat{n}_j \geq 2$ since $\Gamma_j(2) < \bar{\Gamma}_j(2) = \Gamma_j(1) + \Gamma_j(2)$. First we verify that $N_j^+ = \hat{n}_j$ yields a Nash equilibrium. For $m \leq N_j^+ = \hat{n}_j$ we have $\Gamma_j(m) < \bar{\Gamma}_j\left(N_j^+\right)$ as

$$\Gamma_j(m) \leq \Gamma_j\left(N_j^+\right) < \bar{\Gamma}_j\left(N_j^+\right).$$

For $m > N_j^+ = \hat{n}_j$ we have from Lemma 2 that

$$\Gamma_j(m) \geq \bar{\Gamma}_j(m) \geq \bar{\Gamma}_j\left(N_j^+\right)$$

and so $Q_j(m) = 0$ maximizes profits for $m > N_j^+$. It follows that there is a Nash equilibrium with $N_j^+ = \hat{n}_j$.

We now show that $N_j^+ < \hat{n}_j$ cannot lead to a Nash equilibrium. In this case there is an m with $Q_j(m) = 0$ for $N_j^+ < m \leq \hat{n}_j$. But $\Gamma_j(m) < \bar{\Gamma}_j\left(N_j^+\right)$ since from Lemma 1

$$\Gamma_j(m) < \bar{\Gamma}_j(\hat{n}_j) < \bar{\Gamma}_j\left(N_j^+\right).$$

Thus from (13)

$$\pi_j(m) = v_j \frac{Q_j(m)}{Q_j}\left(1 - \frac{\Gamma_j(m)}{\bar{\Gamma}_j\left(N_j^+\right)}\right) > 0$$

for $Q_j(m) > 0$, and so firm m has an incentive to operate with $Q_j(m) > 0$.

We now show that $N_j^+ > \hat{n}_j$ cannot lead to a Nash equilibrium. In this case $\Gamma_j\left(N_j^+\right) \geq \bar{\Gamma}_j\left(N_j^+\right)$ so

$$s_j\left(N_j^+\right) = 1 - \frac{\Gamma_j\left(N_j^+\right)}{\bar{\Gamma}_j\left(N_j^+\right)} \leq 0$$

which is not consistent with the requirement $Q_j\left(N_j^+\right) > 0$.