Heterogeneity and Optimal Monetary Policy

Michael Sampson
Department of Economics
Concordia University,
1455 de Maisonneuve Blvd.West,
Montreal, Quebec, Canada, H3G 1M8
Email: michael.sampson@concordia.ca

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Abstract

Monetary policy is examined in a model with heterogeneous individuals and two types of money: inside and outside money. A closed-form solution is derived with a simple formula for each individual's optimal monetary rule. This formula is used to explain the political forces the government faces in adopting a monetary rule; for example why the Friedman rule would not be adopted despite being Pareto optimal. When the model is calibrated to U.S. data, it is shown that the very poor favor the Friedman rule, but the remaining population have an overwhelming incentive to block its implementation.

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1 Introduction

Friedman (1969) argued that optimal monetary policy is characterized by a nominal interest rate of $R = 0$, the so-called Friedman rule. Later Phelps (1973) showed that optimal monetary policy is an optimal taxation problem where the inflation tax R is just one of many distorting taxes. Using the Ramsey rule, Phelps argued for $R > 0$. More recently optimal monetary policy has been sought within the framework of dynamic general equilibrium models. Examples include Kimbrough (1986), Faig (1988), Woodford (1990), Correia and Teles (1996), Chari, Christiano, Kehoe (1996), da Costa and Werning (2003), and Bhattacharya, Haslag, and Martin (2005). In some models the Friedman rule $R = 0$ is optimal while in others $R > 0$ is optimal. Casey and Sala-i-Martin

(1997) survey this literature and provide a nice summary of the controversy, as well as an attempt at a unified framework. Finding no clear theoretical answer they, along with Dotsey and Ireland (1996), attempt to empirically estimate the optimal R.

Much of the literature assumes a representative agent, but recently interest has focused on the implications of heterogeneity for optimal monetary policy (see for example: Erosa and Ventura, 2002, da Costa and Werning, 2003, and Bhattacharya, Haslag, and Martin, 2005). With heterogeneity there is not a single optimal monetary rule, but instead a continuum of Pareto optimal R, a contract curve. Which R on the contract curve the government chooses is then a political decision. In this paper we seek to understand the political forces that shape this decision by characterizing the types of individuals who favor particular monetary rules, all the way from the Friedman rule $R = 0$ to the hyperinflation rule $R = \infty$. To this end we develop a dynamic general equilibrium model with an arbitrary number of heterogeneous individuals, a government with its own utility function, and two types of money: inside money (cash) and outside money (cheques).

A distinguishing feature of our model is that it has a closed-form solution. In particular we are able to derive a simple expression for the optimal monetary rule R_i^* for each individual i in the economy, which in turn can be used to understand the political forces that determine the actual R chosen by the government.

Of course a closed-form solution requires strong assumptions, but all models that generate interesting results require strong assumptions, and in any case models in this literature cannot credibly be thought of as structural representations of the actual data generating process. Instead one works with stylized models meant to provide insight, educated guesses about orders of magnitude, and to suggest further lines of inquiry.

Even though our model is stylized, it is realistic enough that it can be matched to U.S. aggregate time series, and the disaggregate panel data found in Avery, Elliehausen, Kennickell, and Spindt (1986). Consequently we are able to use our model to make an educated guess about what the optimal monetary rule is for various income groups in the United States. Our results suggest that the poor, who use more cash, will want monetary rules that are closer to the Friedman rule than the rich, a result similar to Erosa and Ventura (2002), who use a very different setup. Our model predicts that Friedman rule $R = 0$ will be optimal for individuals in the economy who use only cash to finance their transactions, about 15% of the population, mostly the very poor and uneducated. For the remaining 85% of the population the optimal monetary rule is very close to the Friedman rule, but we show that actually adopting the Friedman rule would generate unbounded welfare losses for the remaining 85% of the population. So our model provides an explanation why in a democracy the Friedman rule would not be adopted despite being Pareto optimal.

2 The Model

Here we present the details of the model. Readers interested primarily in the results can skip to equation (16) on page 7 where the optimal monetary rule R_i^* is given, along with a summary of the important notation.

We consider a dynamic competitive exchange economy. Time $t = 0, 1, 2, \ldots$ is discrete with $t = 0$ the initial period. There are $i = 1, 2, \ldots, n$ heterogeneous private individuals and a selfish government indexed as g. When the n private individuals are aggregated into a representative individual we replace i with p. Before trading takes place in any period t we use ˜ over any variable to indicate an endowment, and the same variable without a ˜ to indicate final demand. For example $\tilde{C}_{it}$ is individual $i's$ endowment of the consumption good at time t, C_{it} is individual $i's$ final demand, and $C_{pt} \equiv \sum_{i=1}^{n} C_{it}$ is the representative individual's final demand.

There are three types of assets: 1) physical capital, 2) inside money, which we refer to as cash, and 3) outside money which we refer to as checks. Physical capital has a total supply K_t and price P_t^k, cash has a total supply M_t^0 and price P_t^0, and checks have a total supply M_t^1 and price P_t^1. Each unit of each asset produces, without any labour input, like fruit dropping off a tree, one unit of a good or service that can be traded, and which enters into the utility of private individuals and government.

Physical capital K_t produces one unit of a perishable consumption good with total supply $C_t = K_t$ and price P_t^c. The two types of money each provide a service, which can be thought of as facilitating exchange in some manner. One unit of cash M_t^0 produces one unit of cash services with total supply $S_t^0 = M_t^0$ and price of cash services R_t^0. Cash services can be thought of as facilitating small or anonymous transactions where money theft is not an issue. One unit of checks M_t^1 produces one unit of check services with total supply $S_t^1 = M_t^1$ and price of check services R_t^1. Check services can be thought of as facilitating large transactions where anonymity is not important, but money theft is.

Our approach to money is in the spirit of the money-in-the-utility-function literature, but with an important difference: it is not money itself but the *services* of money that enters the utility function. Although this may at first appear novel or controversial, it is really quite standard: individuals do not get utility directly from an asset, but instead from the goods or services that the asset provides.

The role of money in our model is no more mysterious than that of a capital good that acts as a mode of transportation, like a car. Like money a car provides a service by making transactions easier in a particular way: say buying groceries at a distant mall. It is these services that go into the utility function, and not the car or money itself. It is the prospect of future services that make cars and money valuable as assets. Like money in our model, in the real world there is a separate market for car services (the car rental market) where cars are lent on a short-term basis. Having a separate market for the services of money or cars allows individuals to enjoy the services of money or a car without necessarily owning money or a car. And just as cars and trucks facilitate transactions in different ways, cash and cheques in our model facilitate transactions in different ways, each with its own advantages and disadvantages. What ultimately makes money different than a car in our model is that it is produced at no cost by the government.

The markets for money services then can be thought of as short-term lending markets where owners of money lend their money, and hence the corresponding money services, for the duration of the trading period at time t. The notation R_t^0, R_t^1 for the prices of money services is used to suggest short-term interest rates. Because there are separate markets for money and money services, the role of medium of exchange can be separated from the role of store of value.

Only private individuals (and not the government) hold capital so $K_t = \sum_{i=1}^n K_{it}$ where K_{it} is individual $i's$ demand for capital. Both individuals and the government demand the consumption good C_t, with individual $i's$ demand denoted by C_{it}, the government's demand denoted by C_{gt}, and the representative individuals's demand given by $C_{pt} = \sum_{i=1}^n C_{it}$ so that $C_t = C_{pt} + C_{gt}$. We could have had capital K_t and hence consumption grow at a rate ρ as $K_t = C_t = (1+\rho) K_{t-1}$, but none of our results depend on ρ, and so we simplify by setting $\rho = 0$ so that

$$K_t = C_t = K_{t-1},\ \tilde{K}_{it} = \tilde{C}_{it} = K_{it-1}. \tag{1}$$

In the initial period $t = 0$, each individual i owns the same proportion σ_i of the each of the three assets with $\sum_{i=1}^n \sigma_i = 1$ so that at $t = 0$

$$\tilde{K}_{it} = \sigma_i K_t, \tilde{M}_{it}^0 = \sigma_i M_t^0, \tilde{M}_{it}^1 = \sigma_i M_t^1. \tag{2}$$

In fact we will show that (2) holds for all t, and that the weights σ_i can be used to construct a representative individual that explains all prices and aggregate quantities.

Cash acts as a numeraire so $P_t^0 = 1$. The government, acting as the economy's banker, stands prepared to exchange one unit of cash for one unit of checks so that $P_t^1 = 1$. This leads to an endogenously determined money multiplier θ as $M_t^1 = \theta \times M_t^0$.

Only private individuals hold money or use money services and so $M_t^0 = \sum_{i=1}^n M_{it}^0 = \sum_{i=1}^n S_{it}^0$ and $M_t^1 = \sum_{i=1}^n M_{it}^1 = \sum_{i=1}^n S_{it}^1$ (where M_{it}^0, M_{it}^1 are individual i' demand for cash and cheques, and S_{it}^0, S_{it}^1 are the corresponding demand for cash and cheque services). The government is required to fix the growth rate of cash τ_m for all periods (and so we rule out issues commitment and time inconsistency as considered for example by Lucas and Stokey, 1983) so that $S_t^0 = M_t^0 = (1+\tau_m) M_{t-1}^0$. The amount of cash printed in each period, $\tau_m M_{t-1}^0$ and the services from this cash $\tau_m S_{t-1}^0$ are used by the government to finance its expenditures.

If $\tilde{M}_{it}^0$ and $\tilde{S}_{it}^0$ are individual $i's$ endowments of cash services before trading then

$$\tilde{S}_{it}^0 = \tilde{M}_{it}^0 = M_{it-1}^0. \tag{3}$$

Since the government insures that cash and checks trade one-for-one as $P_t^1 = 1$, checks must also grow at the same rate τ_m as cash. But unlike cash growth that is kept by the government as seigniorage, check growth is kept by private individuals. Thus if $\tilde{M}_{it}^{1}$ and $\tilde{S}_{it}^{1}$ are individual $i's$ endowments of checks and check services, then

$$\tilde{S}_{it}^{1} = \tilde{M}_{it}^{1} = (1+\tau_m)\, M_{it-1}^{1}. \qquad (4)$$

Utility for individual i is assumed to depend on consumption C_{it}, the real value of cash services $\frac{S_{it}^{0}}{P_t^c}$, and the real value of check services $\frac{S_{it}^{1}}{P_t^c}$ as

$$U_{it} = \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \left(\alpha_i^c \ln (C_{it+k}) + \alpha_i^0 \ln \left(\frac{S_{it+k}^{0}}{P_{t+k}^{c}} \right) + \alpha_i^1 \ln \left(\frac{S_{it+k}^{1}}{P_{t+k}^{c}} \right) \right) \qquad (5)$$

where $r > 0$ is the discount factor, $\alpha_i^c > 0, \alpha_i^0 \geq 0, \alpha_i^1 \geq 0$, and $\alpha_i^c + \alpha_i^0 + \alpha_i^1 = 1$. In addition we assume $\alpha_i^0 > 0$ for some i so that at least one individual values the services of cash.

Although the model does not require this, it is useful when anticipating later empirical results (see Table 4 in the empirical section) to think of the relative importance individual i attaches to check and cash services, as given by $\phi_i \equiv \frac{\alpha_i^1}{\alpha_i^0}$, as being an increasing function of σ_i as $\phi_i(\sigma_i)$ with $\phi_i'(\sigma_i) > 0$. That is, wealthier individuals with a high σ_i will want to make relatively more check transactions than cash transactions since for larger transactions money theft is more of an issue.

At $t = 0$ the government imposes a uniform income tax τ on all private income from goods and services, including the services of money, so that households face the wealth constraint

$$\begin{aligned} W_{it} &= P_t^c \tilde{C}_{it} \times (1-\tau) + P_t^k \tilde{K}_{it} + R_t^0 \times \tilde{S}_{it}^0 \times (1-\tau) + \tilde{M}_{it}^0 + R_t^1 \times \tilde{S}_{it}^1 \times (1-\tau) + P_t^1 \tilde{M}_{it}^1 \\ &= P_t^c C_{it} + P_t^k K_{it} + R_t^0 S_{it}^0 + P_t^0 M_{it}^0 + R_t^1 S_{it}^1 + P_t^1 M_{it}^1. \qquad (6) \end{aligned}$$

We assume a selfish government that is only interested in its share of the consumption good C_{gt} with utility function at time t given by

$$U_{gt} = \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \ln (C_{gt+k}). \qquad (7)$$

Since the government cannot hold capital or money across periods, maximizing (7) amounts to maximizing C_{gt} in each period and so nothing depends on the assumption of log utility. The government finances its expenditures with a flat rate income tax rate τ and by printing cash at a rate τ_m so that

$$P_t^c C_{gt} = \tau \sum_{i=1}^{n} \left(P_t^c \tilde{C}_{it} + R_t^0 \tilde{S}_{it}^0 + R_t^1 \tilde{S}_{it}^1 \right) + \tau_m M_{t-1}^0 + \tau_m R_t^0 \times S_{t-1}^0. \qquad (8)$$

3 The Competitive Solution

We now describe the competitive solution conditional on a given monetary policy τ_m and fiscal policy τ. Proofs have been put in the Appendix.

The competitive solution maintains the pattern of wealth ownership given by σ_i in (2) for all t. With the wealth shares σ_i for $i = 1, 2, \ldots, n$ constant over time, we construct a representative individual with utility U_{pt} as

$$U_{pt} = \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \left(\alpha_p^c \ln (C_{pt+k}) + \alpha_p^0 \ln \left(\frac{S_{t+k}^0}{P_{t+k}^c} \right) + \alpha_p^1 \ln \left(\frac{S_{t+k}^1}{P_{t+k}^c} \right) \right) \tag{9}$$

$$\alpha_p^c \equiv \sum_{i=1}^{n} \sigma_i \alpha_i^c, \; \alpha_p^0 \equiv \sum_{i=1}^{n} \sigma_i \alpha_i^0, \alpha_p^1 \equiv \sum_{i=1}^{n} \sigma_i \alpha_i^1.$$

All prices and aggregate quantities in the model can be explained by assuming this representative individual, and so the heterogeneity of the n individuals plays no essential role in explaining aggregate prices and quantities.

Rather than using τ_m to characterize monetary policy, we use the nominal rate of interest R defined as

$$1 + R \equiv (1 + r)(1 + \tau_m). \tag{10}$$

For any given discount factor r there is a one-for-one relationship between τ_m and R. Note that $R \approx r + \tau_m$ and so we can identify r with the real rate of interest, and a Fisher-type relationship holds. A contracting money supply $\tau_m < 0$ is equivalent to $R < r$, and an expanding money supply $\tau_m > 0$ is equivalent to $R > r$. In equilibrium the nominal returns on all three assets are equalized as

$$\frac{P_{t+1}^c (1-\tau) + P_{t+1}^k}{P_t^k} = \frac{R_{t+1}^0 (1-\tau) + P_{t+1}^0}{P_t^0} = \frac{R_{t+1}^1 (1-\tau) + P_{t+1}^1}{P_t^1} (1 + \tau_m) = 1 + R.$$

Consumption C_{it} of individual i, consumption of the representative individual C_{pt}, and the consumption of the government C_{gt} are then given by

$$C_{it} = \frac{\sigma_i \alpha_i^c}{\alpha_p^c + \gamma} C_t, \; C_{pt} = \frac{\alpha_p^c}{\alpha_p^c + \gamma} C_t, \; C_{gt} = \frac{\gamma}{\alpha_p^c + \gamma} C_t \tag{11}$$

where γ is defined as

$$\gamma \equiv \frac{\tau}{1-\tau} + \alpha_p^0 \left(1 - \frac{r}{R} \right). \tag{12}$$

The parameter γ completely summarizes the government's interests in the choice of monetary policy R and fiscal policy τ, or in other words, the government is indifferent to any R, τ that leads to the same γ.

The price of capital P_t^k and consumption P_t^c are

$$P_t^c = \frac{r}{1-\tau} P_t^k = \frac{R \times (\alpha_p^c + \gamma)}{(1-\tau)\alpha_p^0} \frac{M_t^0}{K_t}, P_t^k = \frac{R \times (\alpha_p^c + \gamma)}{r \alpha_p^0} \frac{M_t^0}{K_t}. \tag{13}$$

The prices cash services R_t^0 and chequing services R_t^1 are constant over time as

$$R_t^0 = \frac{R}{1-\tau}, R_t^1 = \frac{r}{1-\tau}. \tag{14}$$

The requirement that checks have a price of $P_t^1 = 1$ yields a money multiplier $M_t^1 = \theta \times M_t^0$ where $\theta = \frac{\alpha_p^1}{\alpha_p^0} \frac{R}{r}$.

We will need to be able to evaluate the effect of R on the utility U_{it} of individual i. For this it can be shown utility U_{it} for individual i can be expressed as

$$U_{it}\left(R, C_t, \gamma\right) = \frac{1+r}{r}\left(-\ln\left(\alpha_p^c + \gamma\right) - \left(1 - \alpha_i^c\right)\ln\left(1 + \gamma - \alpha_p^0\left(1 - \frac{r}{R}\right)\right) - \alpha_i^0 \ln\left(R\right) + \ln\left(C_t\right)\right). \tag{15}$$

The welfare of the representative individual $U_{pt}\left(R, C_t, \gamma\right)$ is found by replacing i with p in (15) with $\alpha_p^c, \alpha_p^0, \alpha_p^1$ defined in (9).

4 Optimal Monetary Policy

We now seek to characterize the contract curve, the set of Pareto optimal R, τ. When doing this we include the government's utility in our calculations, and so we insure that any R, τ on the contract curve is politically feasible. In other words the government will not have any incentive to block any Pareto improving change in R, τ.

Recall that the government is indifferent to any R, τ that leads to the same γ in (12) with higher (lower) values of γ making the government better (worse) off. As well higher (lower) values of γ make all private individuals worse (better) off since from (15) $\frac{\partial U_{it}}{\partial \gamma} < 0$. So if we condition on a given γ we can ignore government welfare in any subsequent calculations. From (15) it follows that conditional on γ the optimal R for individual i is

$$R_i^* = \frac{\alpha_i^1}{\alpha_i^0} \frac{r\alpha_p^0}{1 + \gamma - \alpha_p^0}. \tag{16}$$

This simple formula gives the optimal nominal interest rate or monetary rule R_i^* for individual i as a function of: 1) the real rate of interest r, 2) the relative importance of cheque services α_i^1 to cash services α_i^0 for individual i as given by $\frac{\alpha_i^1}{\alpha_i^0}$, 3) the importance of cash services for the representative individual as given by α_p^0, and 4) the size of the government in the economy as given by γ. From the expression for γ in (12) the implied optimal tax rate for individual i then is

$$\tau_i^* = 1 - \left(1 + \gamma - \alpha_p^0\left(1 - \frac{r}{R_i^*}\right)\right)^{-1}. \tag{17}$$

Even though R_i^* is derived from a stylized model and so cannot be literally true, we believe it is a useful formula to begin thinking about optimal monetary policy when there is heterogeneity, just as the Friedman rule $R = 0$ can be thought of as a useful beginning for thinking about optimal monetary policy without heterogeneity.

Here R_i^*, τ_i^* is Pareto optimal since $\frac{\partial U_{it}}{\partial R} < 0$ for $R > R_i^*$ and $\frac{\partial U_{it}}{\partial R} > 0$ for $R < R_i^*$. Any movement of R towards R_i^* makes individual i better off, and any movement of R away from R_i^* makes individual i worse off. It follows that R_i^*, τ_i^* is a point on the contract curve, and that conditional on γ, individual i prefers R_i^*, τ_i^* to any other point R, τ on the contract curve. To find the entire contract curve define $R_{\min}^* \equiv \min_i \; R_i^*$, $R_{\max}^* \equiv \max_i \; R_i^*$. Conditional on γ the contract curve is described by $R_{\min}^* \leq R \leq R_{\max}^*$ with the implied τ determined from (12). The range of the contract curve $R_{\max}^* - R_{\min}^*$ will depend on the heterogeneity of tastes, in particular regarding α_i^0 and α_i^1.

4.1 The Friedman rule

The Friedman rule will be a Pareto optimal monetary rule if $R_i^* = 0$ for some individual i.[1] From (16) this will occur if this individual i uses only cash (and not checks) to finance his transactions. This does not preclude other monetary rules $R > 0$ from also being Pareto optimal. But if all individuals in the economy use only cash, then the Friedman rule is the unique Pareto optimal monetary rule.

Theorem 1 ***Pareto Optimality of the Friedman Rule:*** *If $\alpha_i^0 > 0$ and $\alpha_i^1 = 0$ for some i, then the Friedman rule $R = 0$ is Pareto optimal.*

Theorem 2 ***Uniqueness of the Friedman Rule:*** *If $\alpha_i^1 = 0$ for all i (individuals use only cash), then the Friedman rule $R = 0$ is the unique Pareto optimal monetary rule.*

If nobody uses checks or $\alpha_i^1 = 0$ for all i, then this is equivalent to a model with only one form of money: cash. So the existence of outside money in our model is critical to whether or not the Friedman rule $R = 0$ is uniquely optimal.

According to Avery, Elliehausen, Kennickell, and Spindt (1986), about 15 percent of the U.S. population, mostly the poor, use only cash in their transactions and so have $\alpha_i^1 = 0$ and $R_i^* = 0$, and so, according to the model, favor the Friedman rule $R = 0$.

We now show that those who use checks and so have $\alpha_i^1 > 0$ and $R_i^* > 0$ (the remaining 85 percent of the U.S. population) have an overwhelming incentive block implementation of the Friedman rule. To this end consider measuring the welfare gain for individual i, with γ held fixed, from moving from one monetary regime R_1 to another monetary regime R_2 . Define $\delta_i\left(R_2, R_1\right)$ as the proportionate change in each period's consumption needed to make individual i indifferent between R_1 and R_2 as

$$U_{it}\left(R_1, C_t, \gamma\right) \equiv U_{it}\left(R_2, e^{\delta_i(R_2,R_1)} C_t, \gamma\right).$$

From (15) it follows that

$$\delta_i\left(R_2, R_1\right) = -\alpha_i^1 \ln\left(\frac{R_2}{R_1}\right) + \left(1 - \alpha_i^c\right) \ln\left(\frac{\left(1 + \gamma - \alpha_p^0\right) R_2 + \alpha_p^0 r}{\left(1 + \gamma - \alpha_p^0\right) R_1 + \alpha_p^0 r}\right). \tag{18}$$

[1] The competitive equilibrium actually requires $R > 0$, but $R_i^* = 0$ is still meaningful: for any $\hat{R} > 0$ the government can always increase the welfare of individual i by choosing an $0 < R < \hat{R}$ arbitrarily close to $R_i^* = 0$.

From the term $-\alpha_i^1 \ln\left(\frac{R_2}{R_1}\right)$ in (18) it follows that check users with $\alpha_i^1 > 0$ experience an unbounded welfare loss as the Friedman rule is approached from above.

Theorem 3 ***Infinite Welfare Loss from the Friedman Rule:*** *If $R_1 > 0$ is the status quo interest rate and $\alpha_i^1 > 0$, then $\lim_{R_2 \downarrow 0} \delta_i (R_2, R_1) = \infty$.*

Adopting the Friedman rule $R = 0$ has other drastic implications. First, adopting the Friedman rule implies 100% taxation or $\tau = 1$ since from (12) we have as $R \downarrow 0$ that $\tau \uparrow 1$. This happens because $R = 0$ implies negative seigniorage, and so to keep γ fixed and the government happy, the tax rate τ must be continually increased as $R \downarrow 0$. Second, adopting the Friedman rule drives the real and nominal price of capital to zero (a stock market crash). From (13) as $R \downarrow 0$ we have for a fixed γ that

$$\lim_{R \downarrow 0} P_t^k = \lim_{R \downarrow 0} \frac{R \times \left(\alpha_p^c + \gamma\right)}{r \alpha_p^0} \frac{M_t^0}{K_t} = 0.$$

Thus the nominal price of capital approaches 0. The real price of capital defined by

$$\pi_k \equiv \frac{P_t^k}{P_t^c} = \frac{1-\tau}{r} \tag{19}$$

also approaches 0 as $R \downarrow 0$ since $\tau \uparrow 1$.

So from the point of view of the model it is easy to see why the Friedman rule is not politically feasible. It would be optimal for 15% of the American population who use only cash, but it would be very undesirable for the remaining 85% of the American population.

4.2 Optimal Monetary Policy for the Representative Individual

Now suppose we attempt to arrive at a single optimal monetary policy rule by basing our calculations on the representative individual with utility function U_{pt} in (9). By setting $\alpha_i^0 = \alpha_p^0$ and $\alpha_i^1 = \alpha_p^1$ in (16), the optimal interest rate R_p^* and tax rate τ_p^* (conditional on γ) for the representative individual is

$$R_p^* = \frac{r \alpha_p^1}{1 + \gamma - \alpha_p^0}, \tau_p^* = 1 - \left(1 + \gamma - \alpha_p^0 \left(1 - \frac{r}{R_p^*}\right)\right)^{-1}.$$

As long as one individual values the services of checks it will be the case that $\alpha_p^1 > 0$, and so $R_p^* > 0$. So if we had started by assuming a representative individual, we would have concluded that the Friedman rule $R = 0$ is not optimal, even though $R_p^* > 0$ does not contradict the Friedman rule $R = 0$ being Pareto optimal, which only requires that there is one individual who uses only cash.

Even though the Friedman rule is not optimal for the representative individual, we can still obtain a weak form of the Friedman rule: optimal monetary policy for the representative individual implies a contracting money supply or $\tau_m < 0$ since

$$R_p^* = r \times \frac{\alpha_p^1}{\alpha_p^1 + \alpha_p^c + \gamma} < r.$$

Theorem 4 ***Weak Friedman Rule:*** *For the representative individual $R_p^* < r$, or equivalently $\tau_m < 0$.*

In fact for realistic parameter values we can expect R_p^* to be very close to the Friedman rule or $R_p^* \approx 0$. This is because r is small, say of the order of $r = 0.02$, the wealth share of checks α_p^1 is small, much less than 1, and with a large government $1 + \gamma - \alpha_p^0 > 1$. For example when we later match the model to U.S. data we obtain $R_p^* = 0.00041$.

Although we can expect $R_p^* \approx 0$, it does not follow that simply rounding R_p^* off to the Friedman rule of $R = 0$ will be good enough. The logic of the infinite welfare loss result (Theorem 3) still holds, adopting the Friedman rule $R = 0$ causes an infinite welfare loss to anyone who uses checks, including the representative individual; that is $\lim_{R \downarrow 0} \delta_p \left(R, R_p^*\right) \to \infty$. This illustrates the extreme sensitivity of the economy to small changes in R in the regime around the weak Friedman rule $R_p^* = 0$.

4.3 How Well Does the Representative Individual Represent?

The Friedman rule $R = 0$ is optimal for those individuals who only use cash ($\alpha_i^1 = 0$), but leads to an infinite welfare loss for those who use checks ($\alpha_i^1 = 0$). But a weak version of the Friedman rule $0 < R_p^* < r$ with a contracting money supply $\tau_m < 0$ applies to the representative individual. So the model can still give a unique policy recommendation that resembles the Friedman rule if we are willing to rely on the representative individual. But how well does the representative individual represent the welfare of individuals in the economy? To put the matter more precisely: does the upper bound r (or contracting money supply $\tau_m < 0$) of the weak Friedman rule $0 < R_p^* < r$ necessarily characterize the entire contract curve?

The answer is no. To see this we have from (16) that

$$R_i^* = \frac{\phi_i}{\phi_p} R_p^* \tag{20}$$

where $\phi_i \equiv \frac{\alpha_i^1}{\alpha_i^0}$ measures the relative importance of check and money services for individual i. Thus $R_i^* \lesseqqgtr R_p^*$ as $\phi_i \lesseqqgtr \phi_p$. But given enough heterogeneity ϕ_i can take on any positive value, (i.e., $0 \leq \phi_i \leq \infty$) and so there is nothing to rule out $R_i^* > r$ or $\tau_m > 0$. The fact that the real rate of interest r bounds

R_p^* for the representative individual does not imply that r bounds R_i^* for all individuals.

In fact the contract curve has no finite upper bound. The limiting case is where there are individuals who use only checks and no cash or $\alpha_i^1 > 0, \alpha_i^0 = 0$. In this case $\phi_i = \infty$ and so $R_i^* = \infty$. Individuals who use only checks favor a monetary rule that leads to hyperinflation.

Theorem 5 ***Pareto Optimality of Hyperinflation:*** *If $\alpha_i^0 = 0$ and $\alpha_i^1 > 0$ for some i (somebody uses checks exclusively) then $R_i^* = \infty$ is Pareto optimal.*

According to Avery, Elliehausen, Kennickell, and Spindt (1986), everybody in their sample used some cash, and so this suggests that empirically $R_{\max}^* < \infty$, or that everyone in the U.S. would be against hyperinflation. But even if there were a small number of individuals for which hyperinflation is Pareto optimal, it is unlikely that we would observe hyperinflation being adopted since it has dire consequences for the remainder of the population.

Theorem 6 ***Infinite Welfare Loss from Hyperinflation:*** *If $R_1 > 0$ is the status quo interest rate and $\alpha_i^0 > 0$, then $\lim_{R_2 \to \infty} \delta_i (R_2, R_1) = \infty$.*

Just as check users have an overwhelming incentive to block implementation of the Friedman rule $R = 0$, cash users have an equally overwhelming incentive to block implementation of a hyperinflation rule $R = \infty$.

5 An Application to the U.S. Economy

We now explore optimal monetary and fiscal policy when the parameters of the model are matched to U.S. data. We use aggregate U.S. data to determine the parameters that characterize the representative U.S. individual: $\alpha_p^0, \alpha_p^1, \alpha_p^c$, the size of government γ, and the income tax rate τ. We then use the disaggregate U.S. data from Avery *et al.* (1986) to explore the implications of heterogeneity for optimal monetary policy.

We begin by matching the aggregate parameters of the model to a U.S. representative individual using U.S. aggregate annual data for the period 1990 to 2001. To this end define the consumption cash velocity v_t^0, the consumption check velocity v_t^1, and the ratio of government expenditure to consumption π_t^g as

$$v_t^0 \equiv \frac{P_t^c \times C_t}{M_t^0}, v_t^1 \equiv \frac{P_t^c \times C_t}{M_t^1}, \pi_t^g \equiv \frac{P_t^c \times C_{gt}}{P_t^c \times C_{pt}}.$$

We match the cash variable in the model M_t^0 with U.S. base money, the check variable in the model M_t^1 with the broad definition of money *M3* net of base money, nominal consumption in the model $P_t^c \times C_t$ with U.S. nominal non-durables and services, and nominal government expenditure in the model $P_t^c \times C_{gt}$ with nominal U.S. government expenditure. The nominal interest rate in the model R is matched with the U.S. one-year treasury bill rate. The discount rate or the real rate of interest is set at $r = 0.02$. From this U.S. data we calculate the corresponding sample means $\bar{v}^0, \bar{v}^1, \bar{\pi}_g, \bar{R}$ given in Table 1.

Using the values in Table 1 we have

$$9.88 = \bar{v}^0 = \frac{R\alpha_p^c}{\alpha_p^0 (1-\tau)}, 0.95 = \bar{v}^1 = \frac{r\alpha_p^c}{\alpha_p^1 (1-\tau)}, 0.32 = \bar{\pi}_g = \frac{\gamma}{\alpha_p^c}. \tag{21}$$

In addition from (12) and (24) we have

$$\alpha_p^0 + \alpha_p^1 + \alpha_p^c = 1, \gamma = \frac{\tau}{1-\tau} + \alpha_p^0 \left(1 - \frac{r}{\bar{R}}\right) \tag{22}$$

and so we have five equations for the five unknowns: $\alpha_p^0, \alpha_p^1, \alpha_p^c, \gamma, \tau$. Solving these equations yields the values shown in Table 2.

From Table 2, (16), and (17) we find the optimal monetary policy rule and tax rate R_p^*, τ_p^* for the U.S. representative individual is $R_p^* = 0.00041, \tau_p^* = 0.39$. As we argued earlier when discussing Theorem 4, the weak Friedman rule, R_p^* is numerically close to the Friedman rule $R = 0$. From (18) the welfare gain for the U.S. representative individual of moving from the status quo $R = 0.057, \tau = 0.23$ to $R_p^* = 0.00041, \tau_p^* = 0.39$ is $\delta_p \left(R, R_p^*\right) = 0.034$ or the equivalent of 3.4% of annual consumption.

Let us now examine the implications for the stock market of moving to R_p^*, τ_p^*. From (19) the real price of capital is $\pi^k \equiv \frac{1-\tau}{r}$, which only depends on the tax rate τ. Moving from the status quo $\tau = 0.023$ to $\tau_p^* = 0.39$ causes the real price of capital to fall from $\pi^k = 48.85$ to $\pi^k = 30.5$, a 38% drop, comparable in size to many real-world stock market crashes.

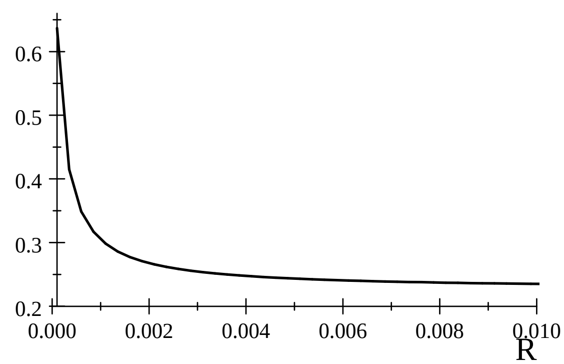
We now turn from the U.S. representative individual to examine the implications of heterogeneity for optimal monetary policy. Avery *et al.* (1986) interviewed 1946 families in the United States to determine their patterns of money use. We summarize the empirical findings relevant for our purposes in Table 3 (derived from their Table 1). We match cash in Table 3 with cash in the model, and checks and credit cards in Table 3 with checks in the model. Consumption expenditure in the model $P_t^c C_{it}$ is decomposed as

$$P_t^c C_{it} = P_t^c C_{it}^0 + P_t^c C_{it}^1$$

where $P_t^c C_{it}^0$ is cash expenditure and $P_t^c C_{it}^1$ is check expenditure. We assume that cash expenditures $P_t^c C_{it}^0$ are proportional to the value of cash services purchased $R_t^0 S_{it}^0$ in the model, and check expenditures $P_t^c C_{it}^1$ are proportional to the value of check services $R_t^1 S_{it}^1$ in the model. The ratio of the two rows in Table 3 then determines ϕ_i, the relative importance of checks to cash for individuals in that income class. For example for individuals making under \$10,000 per year we have from Table 3 that $\phi_i = \frac{47\%}{53\%} \approx 0.9$. This as well as the other values of ϕ_i for the different income groups is reported in Table 4. Table 4 shows that as income increases, ϕ_i increases, that is the rich use relatively more checks than the poor, and the poor use relatively more cash than the rich. From Table 2 the representative individual has $\phi_p \equiv \frac{\alpha_p^1}{\alpha_p^0} = 3.6$, and so the representative individual uses cash and checks in a manner similar to a rather well-off American family having an income in 1986 between \$30,000 and \$50,000.

From (20) and (17) we can calculate the optimal monetary rule R_i^* and tax rate τ_i^* for an individual having an identical ϕ_i as the different income groups in Table 4. This is shown in Table 5. For comparison Table 5 also includes the 15% of the population who use cash only and want the Friedman rule ($\phi_i = 0$), the representative individual with $\phi_p = 3.6$, as well as any individuals who use only checks $\phi_i = \infty$ and want a hyperinflation.

Table 5 associates points on the contract curve with the type of individuals who would desire to be at those points. The contract curve R, τ is then shown in the figure below.



Contract Curve R, τ for $\gamma = 0.30$

The optimal monetary rules R_i^* in Table 5 differ very little from the Friedman rule $R = 0$, even though for any individual with $\phi_i > 0$ a movement to the Friedman rule would cause an infinite welfare loss. But the small numerical differences in R_i^* mask huge differences in attitudes concerning how government policy should be carried out, as can be seen by the implied tax rates, where the differences across individuals are much larger. The poor want to have monetary policies closer to the Friedman rule with very high marginal tax rates, the rich want monetary policies farther away from the Friedman rule with much lower marginal tax rates.

6 Conclusions

There is no reason to believe there is one unique optimal monetary rule: different people will want to be at different points on the contract curve. This paper has explored optimal monetary policy in this spirit. Our results provide a political explanation for why the Friedman rule has never been adopted despite being Pareto optimal. More generally the poor, who use more cash and fewer checks than the rich, will want a more conservative monetary rule, but in return will accept a higher rate of income tax.

The framework we have adopted allows for lots of heterogeneity without sacrificing a closed-form analytical solution. This closed-form solution allowed us to derive a simple expression for the optimal monetary rule for each individual in the economy. An advantage of the framework we adopt is that it can be expanded to allow for much more heterogeneity, and without sacrificing a closed-form solution. In fact the original model we developed had an arbitrary number of consumption goods, capital goods, and types of money. Further possible elaborations include production, a labour/leisure decision, and uncertainty. As long as models are not claimed to be actual representations of a data generating process, in which case they must face much more stringent data tests, then we feel that closed-form solutions should only be abandoned if no alternative exists. The model presented is an attempt to demonstrate how this philosophy might be applied to the optimal money literature.

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8 Appendix

Referring to (6), define wealth shares as

$$\lambda_{it}^{c} \equiv \frac{P_t^c C_{it}}{W_{it}}, \lambda_{it}^{0} \equiv \frac{R_t^0 S_{it}^0}{W_{it}}, \lambda_{it}^{1} \equiv \frac{R_t^1 S_{it}^1}{W_{it}}, \mu_{it}^{k} \equiv \frac{P_t^k K_{it}}{W_{it}}, \mu_{it}^{0} \equiv \frac{M_{it}^0}{W_{it}}, \mu_{it}^{1} \equiv \frac{P_t^1 M_{it}^1}{W_{it}} \tag{23}$$

where

$$\lambda_{it}^{c} + \lambda_{it}^{0} + \lambda_{it}^{1} + \mu_{it}^{k} + \mu_{it}^{0} + \mu_{it}^{1} = 1. \tag{24}$$

Theorem 7 *Individuals choose their wealth shares for consumption, cash services, and check services as* $\lambda_{it}^{c} = \frac{r\alpha_i^c}{1+r}, \lambda_{it}^{0} = \frac{r}{1+r}\alpha_i^0, \lambda_{it}^{1} = \frac{r}{1+r}\alpha_i^1$.

Theorem 8 *Individuals are indifferent to any asset wealth shares* $\mu_{it}^{k}, \mu_{it}^{0}, \mu_{it}^{1}$ *that satisfy* $\mu_{it}^{k} + \mu_{it}^{0} + \mu_{it}^{1} = \frac{1}{1+r}$.

Theorem 9 *Returns on the three assets are equalized as*

$$\frac{P_{t+1}^c (1-\tau) + P_{t+1}^k}{P_t^k} = R_{t+1}^0 (1-\tau) + 1 = \frac{\left(R_{t+1}^1 \times (1-\tau) + P_{t+1}^1\right)}{P_t^1} (1+\tau_m) = 1 + R_{t+1}.$$

Proof. From (1), (3), (4), (6), and (23) we have

$$\begin{aligned} W_{it+1} &= P_{t+1}^c \tilde{C}_{it+1} (1-\tau) + P_{t+1}^k \tilde{K}_{it+1} + R_{t+1}^0 \tilde{S}_{it+1}^0 (1-\tau) + \tilde{M}_{it+1}^0 + R_t^1 \tilde{S}_{it+1}^1 (1-\tau) + P_{t+1}^1 \tilde{M}_{it+1}^1 \\ &= W_{it} \times \Big(\frac{\left(P_{t+1}^c (1-\tau) + P_{t+1}^k\right)}{P_t^k} \mu_{it}^k + \left(R_{t+1}^0 (1-\tau) + 1\right) \mu_{it}^0 \\ &\quad + \frac{\left(R_{t+1}^1 (1-\tau) + P_{t+1}^1\right)}{P_t^1} (1+\tau_m) \mu_{it}^1 \\ &= W_{it} \times (1 + R_{it+1}) \end{aligned}$$

where the one period nominal return for individual i is defined as

$$1 + R_{it+1} \equiv \frac{P_{t+1}^c \times (1-\tau) + P_{t+1}^k}{P_t^k} \mu_{it}^k + \left(R_{t+1}^0 \times (1-\tau) + 1\right) \mu_{it}^0 + \frac{\left(R_{t+1}^1 \times (1-\tau) + P_{t+1}^1\right)}{P_t^1} (1+\tau_m) \mu_{it}^1. \tag{25}$$

Thus

$$W_{it+k} = W_{it} \times \prod_{l=1}^{k} (1 + R_{it+l}). \tag{26}$$

Substituting

$$C_{it+k} = \lambda_{it+k}^c \frac{W_{it+k}}{P_{t+k}^c}, \; S_{it+k}^0 = \lambda_{it+k}^0 \frac{W_{it+k}}{R_{t+k}^0}, S_{it+k}^1 = \lambda_{it+k}^1 \frac{W_{it+k}}{R_{t+k}^1}$$

in the utility function in (15), using (26), and ignoring terms that do not depend on current wealth shares yields

$$U_{it} = \alpha_i^c \ln(\lambda_{it}^c) + \alpha_i^0 \ln\left(\lambda_{it}^0\right) + \alpha_i^1 \ln\left(\lambda_{it}^1\right) + \frac{1}{r} \ln(1 + R_{it+1}).$$

In equilibrium the coefficients $\mu_{it}^k, \mu_{it}^0, \mu_{it}^1$ in $1 + R_{it+1}$ in (25) must all be equal since otherwise nobody would demand one of the assets, which would be inconsistent with an equilibrium. So Theorem 9 follows. Ignoring terms not depending on current period wealth shares we have

$$U_{it} = \alpha_i^c \ln\left(\lambda_{it}^c\right) + \alpha_i^0 \ln\left(\lambda_{it}^0\right) + \alpha_i^1 \ln\left(\lambda_{it}^1\right) + \frac{1}{r} \ln\left(\mu_{it}^k + \mu_{it}^0 + \mu_{it}^1\right).$$

Here U_{it} depends only on the sum $\mu_{it}^k + \mu_{it}^0 + \mu_{it}^1$ and so individuals are indifferent to how the sum is composed. Maximizing U_{it} subject to (24) then yields Theorems 7 and 8. ■

Given Theorem 8 we can seek a competitive equilibrium where all individuals holds the same weights in their portfolios as

$$\mu_{it}^k = \mu_t^k, \mu_{it}^0 = \mu_t^1, \mu_{it}^1 = \mu_t^1. \tag{27}$$

Define the ownership share of each asset as

$$\sigma_{it}^k \equiv \frac{K_{it}}{K_t}, \sigma_{it}^0 \equiv \frac{M_{it}^0}{M_t^0},\ \sigma_{it}^1 \equiv \frac{M_{it}^1}{M_t^1} \tag{28}$$

and total wealth $W_t \equiv \sum_{i=1}^n W_{it}$ and the value of the capital stock $V_t \equiv P_t^k K_t + M_t^0 + P_t^1 M_t^1$.

Theorem 10 *The pattern of capital ownership is fixed for all time t as $\sigma_{it}^k = \sigma_{it}^0 = \sigma_{it}^1 = \sigma_i$.*

Theorem 11 *Shares of nominal wealth are fixed for all time as $W_{it} = \sigma_i W_t$.*

Theorem 12 *Nominal wealth W_t, interest rate R_t, and the value of the capital stock V_{t-1} are related as $W_t = (1 + R_t) V_{t-1}$.*

Theorem 13 $P_t^k K_t = \mu_t^k W_t,\ M_t^0 = \mu_t^0 W_t,\ P_t^1 M_t^1 = \mu_t^1 W_t.$

Proof. By (2) Theorem 10 holds at $t = 0$ so that at $t = 0$ we have $W_{it} = \sigma_i W_t$ and $\sigma_{it}^k = \sigma_{it}^0 = \sigma_{it}^1 = \sigma_i$. Now for a proof by induction assume for $t-1$ that $W_{it-1} = \sigma_i W_{t-1}$ and $\sigma_{it-1}^k = \sigma_i, \sigma_{it-1}^0 = \sigma_i, \sigma_{it-1}^1 = \sigma_i$. Given (27) we have

$$P_t^k K_t \sigma_{it}^k = \mu_t^k W_{it},\ M_t^0 \sigma_{it}^0 = \mu_t^0 W_{it},\ P_t^1 M_t^1 \sigma_{it}^1 = \mu_t^1 W_{it}.$$

From (1), (3), (4), (6), and Theorem 9 we have

$$\begin{aligned}
W_{it} &= P_t^c \tilde{C}_{it} \times (1-\tau) + P_t^k \tilde{K}_{it} + \left(R_t^0 \times \tilde{S}_{it}^0 \times (1-\tau) + \tilde{M}_{it}^0\right) + \left(R_t^1 \times \tilde{S}_{it}^1 \times (1-\tau) + \tilde{M}_{it}^1\right) \\
&= \left(P_t^c \times (1-\tau) + P_t^k\right) K_{it-1} + \left(R_t^0 \times (1-\tau) + 1\right) M_{it-1}^0 \\
&\quad + \left(R_t^1 \times (1-\tau) + P_t^1\right) (1+\tau_m) M_{it-1}^1 \\
&= (1+R_t)\left(P_{t-1}^k K_{it-1} + P_{t-1}^0 M_{it-1}^0 + P_{t-1}^1 M_{it-1}^1\right) \\
&= (1+R_t)\left(P_{t-1}^k \sigma_i K_{t-1} + P_{t-1}^0 \sigma_i M_{t-1}^0 + P_{t-1}^1 \sigma_i M_{it-1}^1\right) \\
&= \sigma_i (1+R_t)\left(P_{t-1}^k K_{t-1} + P_{t-1}^0 M_{t-1}^0 + P_{t-1}^1 M_{t-1}^1\right) = \sigma_i (1+R_t) V_{t-1}
\end{aligned}$$

and so Theorem 11 follows as $W_{it} = \sigma_i (1 + R_t) V_{t-1} = \sigma_i W_t$. Now

$$\begin{aligned}
P_t^k K_t \times \sigma_{it}^k &= P_t^k K_{it} = \mu_t^k W_{it} = \sigma_i \mu_t^k W_t \\
M_t^0 \times \sigma_{it}^0 &= M_{it}^0 = M_t^0 \times \sigma_{it}^0 = \mu_t^0 W_{it} = \mu_t^0 \sigma_i W_t \\
P_t^1 M_t^1 \times \sigma_{it}^1 &= P_t^1 M_{it}^1 = \mu_t^1 W_{it} = \mu_t^1 \sigma_i W_t
\end{aligned}$$

and so summing over i we have Theorem 13. Then Theorem 10 follows as

$$\sigma_{it}^k = \frac{P_t^k K_{it}}{P_t^k K_t} = \frac{\sigma_i \mu_t^k W_t}{\mu_t^k W_t} = \sigma_i, \sigma_{it}^0 = \frac{M_{it}^0}{M_t^0} = \frac{\sigma_i \mu_t^0 W_t}{\mu_t^0 W_t} = \sigma_i, \sigma_{it}^1 = \frac{P_t^1 M_{it}^1}{P_t^1 M_t^1} = \frac{\sigma_i \mu_t^1 W_t}{\mu_t^1 W_t} = \sigma_i.$$

■

Theorem 14 $(1 + r) \times W_t = (1 + R_t) \times W_{t-1}$.

Proof. We have from Theorem 8 that

$$V_t \equiv P_t^k K_t + M_t^0 + P_t^1 M_t^1 = \mu_t^k W_t + \mu_t^0 W_t + \mu_t^1 W_t = W_t \left(\mu_t^k + \mu_t^0 + \mu_t^1\right) = \frac{W_t}{1 + r}$$

and so $V_t = \frac{W_t}{1+r}$. Combining this with Theorem 15 then yields

$$W_t = (1 + R_t) V_{t-1} = (1 + R_t) \frac{W_{t-1}}{1 + r}$$

and so Theorem 14 follows. ■

Theorem 15 $1 + R_t = 1 + R \equiv (1 + r)(1 + \tau_m)$.

Theorem 16 *The share of wealth held as cash is* $\mu_t^0 = \mu^0 \equiv \frac{r\alpha_p^0 (1-\tau)}{R(1+r)}$.

Theorem 17 *The price of cash services is* $R_t^0 = \frac{R}{1-\tau}$.

Theorem 18 *Nominal wealth growths at the same rate as the money supply as* $W_t = (1 + \tau_m) W_{t-1}$.

Proof. From Theorem 9 we have $R_t^0 (1 - \tau) + 1 = 1 + R_t$ and so multiplying both sides by M_t^0 and using $M_t^0 = (1 + \tau_m) M_{t-1}^0$ yields

$$\left(R_t^0 (1 - \tau) + 1\right) M_t^0 = (1 + R_t)(1 + \tau_m) M_{t-1}^0.$$

From Theorems 7 and 11 we have

$$R_t^0 S_t^0 = R_t^0 M_t^0 = \sum_{i=1}^{n} R_t^0 S_{it}^0 = \sum_{i=1}^{n} \frac{r \sigma_i \alpha_i^0}{1 + r} W_t = \frac{r\alpha_p^0}{1 + r} W_t.$$

So from Theorem 14 we have

$$\frac{r\alpha_p^0}{1 + r} W_t (1 - \tau) + \mu_t^0 W_t = (1 + R_t)(1 + \tau_m) \mu_{t-1}^0 W_{t-1} = (1 + r)(1 + \tau_m) \mu_{t-1}^0 W_t.$$

Cancelling W_t from both sides then yields

$$\frac{r\alpha_p^0(1-\tau)}{1+r} + \mu_t^0 = (1+r)(1+\tau_m)\mu_{t-1}^0.$$

From (10) the coefficient on μ_{t-1}^0 is greater than 1 and so this can be solved forward yielding

$$\mu_t^0 = \mu^0 \equiv \frac{r\alpha_p^0(1-\tau)}{((1+r)(1+\tau_m)-1)(1+r)}.$$

From Theorem 13 we then have $M_t^0 = \mu^0 W_t$ and so

$$1+\tau_m = \frac{M_t^0}{M_{t-1}^0} = \frac{\mu^0 W_t}{\mu^0 W_{t-1}} = \frac{W_t}{W_{t-1}}$$

and so Theorem 18 follows. From Theorems 14 and 18

$$(1+r)W_t = (1+r)(1+\tau_m)W_{t-1} = (1+R_t)W_{t-1}$$

and Theorem 15 follows. Using the expression for R in Theorem 15 we can rewrite the expression for μ^0 above as $\mu^0 = \frac{r\alpha_p^0(1-\tau)}{R(1+r)}$ and so Theorem 16 follows. Using $R_t^0(1-\tau)+1 = 1+R$ and solving for R_t^0 then yields Theorem 17. ■

Theorem 19 *The share of wealth held as checks is* $\mu_t^1 = \mu^1 \equiv \frac{\alpha_p^1(1-\tau)}{1+r}$.

Theorem 20 *The check multiplier* $M_t^1 = \theta \times M_t^0$ *needed to insure* $P_t^1 = 1$ *is* $\theta = \frac{\alpha_p^1 R}{\alpha_p^0 r}$.

Theorem 21 *The price of check services is* $R_t^1 = \frac{r}{1-\tau}$.

Proof. From Theorems 9 and 15

$$\frac{\left(R_t^1(1-\tau)+P_t^1\right)\times(1+\tau_m)}{P_{t-1}^1} = 1+R = (1+\tau_m)(1+r).$$

From Theorems 7 and 11 we have

$$R_t^1 S_t^1 = R_t^1 M_t^1 = \sum_{i=1}^n R_t^1 S_{it}^1 = \sum_{i=1}^n \frac{r\sigma_i\alpha_i^1}{1+r}W_t = \frac{r\alpha_p^1}{1+r}W_t$$

and so multiplying both sides of the previous equation by M_{t-1}^1 yields

$$\frac{\left(R_t^1(1-\tau)+P_t^1\right)}{P_{t-1}^1}\times(1+\tau_m)M_{t-1}^1 = (1+\tau_m)(1+r)\times M_{t-1}^1.$$

From

$$\left(R_t^1(1-\tau)+P_t^1\right)M_t^1 = (1+\tau_m)(1+r)\times P_{t-1}^1 M_{t-1}^1$$

it follows that

$$\left(R_t^1(1-\tau)+P_t^1\right)M_t^1 = R_t^1 M_t^1(1-\tau)+P_t^1 M_t^1 = R_t^1 S_t^1(1-\tau)+P_t^1 M_t^1 = \frac{r\alpha_p^1}{1+r}(1-\tau)W_t+\mu_t^1 W_t$$

and so

$$\frac{r\alpha_p^1}{1+r}(1-\tau)W_t+\mu_t^1 W_t = (1+\tau_m)(1+r)\mu_{t-1}^1 W_{t-1} = (1+r)\mu_{t-1}^1 W_t$$

since $W_t = (1+\tau_m)W_{t-1}$ by Theorem 18. Now by cancelling W_t we have

$$\frac{r\alpha_p^1(1-\tau)}{1+r}+\mu_t^1 = (1+r)\mu_{t-1}^1.$$

Using $1+r>1$ and solving forward yields Theorem 19.

Since the government exchanges checks and cash one-for-one we have $P_t^1 = 1$ and $P_t^1 \times M_t^1 = \mu^1 W_t = \frac{\mu^1}{\mu_0}M_t^0$ where $M_t^1 = \theta M_t^0$ from Theorem 13. Thus the money multiplier $M_t^1 = \theta \times M_t^0$ is $\theta = \frac{\mu^1}{\mu_0} = \frac{\alpha_p^1 R}{\alpha_p^0 r}$. Since $P_t^1 = 1$ we have from Theorems 9 and 15 that

$$\left(R_t^1(1-\tau)+1\right)\times(1+\tau_m) = 1+R = (1+r)(1+\tau_m)$$

and so solving for R_1^1 yields Theorem 21. ■

Theorem 22 *The share of wealth held as capital is* $\mu_t^k = \mu^k \equiv \frac{(1-\tau)\left(\alpha_p^c+\gamma\right)}{1+r}$.

Theorem 23 *The price of capital is* $P_t^k = \frac{\mu^k}{\mu^0}\frac{M_t^0}{K_t} = \frac{R\left(\alpha_p^c+\gamma\right)}{r\alpha_p^0}\frac{M_t^0}{K_t}$.

Theorem 24 *The price of consumption* C_t *is* $P_t^c = \frac{r}{1-\tau}P_t^k = \frac{R\left(\alpha_p^c+\gamma\right)}{(1-\tau)\alpha_p^0}\frac{M_t^0}{C_t}$.

Proof. We have from Theorem 8 that $\mu^k = \frac{1}{1+r} - \left(\mu^0+\mu^1\right)$ so using (12), and Theorems 16 and 20 yields Theorem 22. For the price of capital we have from Theorem 13 that $M_t^0 = \mu_0 W_t$ so

$$P_t^k = \mu^k\frac{W_t}{K_t} = \frac{\mu^k}{\mu^0}\frac{M_t^0}{K_t} = \frac{\frac{1-\tau}{1+r}\times\left(\alpha_p^c+\gamma\right)}{\frac{r\alpha_p^0(1-\tau)}{R(1+r)}}\frac{M_t^0}{K_t} = \frac{R\left(\alpha_p^c+\gamma\right)}{r\alpha_p^0}\frac{M_t^0}{K_t}$$

and so P_t^k in Theorem 23 follows. From Theorem 22 it then follows that

$$P_t^k = \frac{\mu^k}{\mu^0}\frac{M_t^0}{K_t} = \frac{\mu^k}{\mu^0}\frac{1+\tau_m}{1+r}\frac{M_{t-1}^0}{K_{t-1}} = \frac{1+\tau_m}{1+r}P_{t-1}^k.$$

Now from Theorems 9, 16, and 22 we have

$$P_t^c(1-\tau)+P_t^k = (1+r)(1+\tau_m)P_{t-1}^k$$

and so

$$P_t^c(1-\tau)+P_t^k = (1+r)(1+\tau_m)P_{t-1}^k = (1+r)P_t^k.$$

Solving for P_t^c yields Theorem 24. ■

Theorem 25 $C_{it} = \frac{\sigma_i \alpha_i^c}{\alpha_p^c + \gamma} C_t, C_{pt} = \frac{\alpha_p^c}{\alpha_p^c + \gamma} C_t,\ C_{gt} = \frac{\gamma}{\alpha_p^c + \gamma} C_t.$

Proof. From

$$P_t^c C_{it} = \frac{r\alpha_i^c}{1+r} W_{it} = \frac{r\alpha_i^c}{1+r} \sigma_i W_t = \frac{r\sigma_i \alpha_i^c}{1+r} \frac{M_t^0}{\mu^0}$$

we have

$$C_{it} = \frac{r\sigma_i \alpha_i^c}{(1+r)\,\mu^0} \frac{M_t^0}{P_t^c} = \frac{r\sigma_i \alpha_i^c}{(1+r)\,\mu^0} \frac{M_t^0}{\frac{r}{1-\tau} \frac{\mu^k}{\mu^0} \frac{M_t^0}{K_t}} = \frac{(1-\tau)\,\sigma_i \alpha_i^c}{(1+r)\,\mu^k} K_t = \frac{(1-\tau)\,\sigma_i \alpha_i^c}{(1+r)\,\mu^k} C_t = \frac{\sigma_i \alpha_i^c}{\alpha_p^c + \gamma} C_t$$

so that

$$C_{pt} = \sum_{i=1}^{n} \frac{\sigma_i \alpha_i^c}{\alpha_p^c + \gamma} C_t = \frac{\alpha_p^c}{\alpha_p^c + \gamma} C_t$$

which from $C_t = C_{pt} + C_{gt}$ yields $C_{gt} = \frac{\gamma}{\alpha_p^c + \gamma} C_t$. ■

Theorem 26 *The real value of money services is*

$$\frac{S_{it}^0}{P_t^c} = \frac{\sigma_i \alpha_i^0 (1-\tau)}{Rr\left(\alpha_p^c + \gamma\right)} C_t,\ \frac{S_{it}^1}{P_t^c} = \frac{\sigma_i \alpha_i^1 (1-\tau)}{r^2\left(\alpha_p^c + \gamma\right)} C_t.$$

Proof. From Theorems 7, 15, and 17 we have

$$S_{it}^0 = \frac{r\alpha_i^0}{1+r} \frac{W_{it}}{R_t^0} = \frac{r\sigma_i \alpha_i^0}{1+r} \frac{W_t}{\frac{R}{1-\tau}} = \frac{\sigma_i \alpha_i^0 (1-\tau)}{(1+r)\,R} W_t$$

and so using Theorem 23

$$\frac{S_{it}^0}{P_t^c} = \frac{\sigma_i \alpha_i^0 (1-\tau)}{(1+r)\,R} \frac{W_t}{P_t^c} = \frac{\sigma_i \alpha_i^0 (1-\tau)}{(1+r)\,R} \frac{W_t}{\frac{r}{1+r} \times \left(\alpha_p^c + \gamma\right) \frac{W_t}{C_t}} = \frac{\sigma_i \alpha_i^0 (1-\tau)}{Rr\left(\alpha_p^c + \gamma\right)} C_t.$$

Now from Theorems 7, 15, and 21 we have

$$S_{it}^1 = \frac{r\alpha_i^1}{1+r} \frac{W_{it}}{R_t^1} = \frac{r\sigma_i \alpha_i^1}{1+r} \frac{W_{it}}{\frac{r}{1-\tau}} = \frac{\sigma_i \alpha_i^1 (1-\tau)}{r\,(1+r)} W_t$$

and so from Theorem 24 we have

$$\frac{S_{it}^1}{P_t^c} = \frac{\sigma_i \alpha_i^1 (1-\tau)}{r\,(1+r)} \frac{W_t}{P_t^c} = \frac{\sigma_i \alpha_i^1 (1-\tau)}{r\,(1+r)} \frac{W_t}{\frac{r}{1+r} \times \left(\alpha_p^c + \gamma\right) \frac{W_t}{C_t}} = \frac{\sigma_i \alpha_i^1 (1-\tau)}{r^2\left(\alpha_p^c + \gamma\right)} C_t.$$

■

Theorem 27 *For each individual*

$$U_{it}(R, C_t) = \frac{1+r}{r} \left(-\ln\left(\alpha_p^c + \gamma\right) - f_i(R) + \ln(C_t)\right)$$

where

$$f_i(R) = (1-\alpha_i^c) \ln\left(1 + \gamma - \alpha_p^0 \left(1 - \frac{r}{R}\right)\right) + \alpha_i^0 \ln(R)\,. \tag{29}$$

Proof. Putting the result of Theorem 26 in the utility function in (15) yields

$$\begin{aligned} U_{it} &= \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \left(\alpha_i^c \ln (C_{it+k}) + \alpha_i^0 \ln \left(\frac{S_{it+k}^0}{P_{t+k}^c} \right) + \alpha_i^1 \ln \left(\frac{S_{it+k}^1}{P_{t+k}^c} \right) \right) \\ &= \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \left(\alpha_i^c \ln \left(\frac{\sigma_i \alpha_i^c}{\alpha_p^c + \gamma} C_t \right) + \alpha_i^0 \ln \left(\frac{\sigma_i \alpha_i^0 (1-\tau)}{Rr (\alpha_p^c + \gamma)} C_t \right) + \alpha_i^1 \ln \frac{\sigma_i \alpha_i^1 (1-\tau)}{r^2 (\alpha_p^c + \gamma)} C_t \right). \end{aligned}$$

Ignoring terms that do not depend on R, γ or C_t, using $\alpha_i^c + \alpha_i^0 + \alpha_i^1 = 1$ and (12) yields Theorem 27. ■

Theorem 28 $R_i^* = \frac{r\alpha_p^0 \alpha_i^1}{\alpha_i^0 (1+\gamma-\alpha_p^0)}$. *Any movement of R towards R_i^* makes individual i better off, any movement of R away from R_i^* makes i worse off.*

Proof. From (29) we have

$$f_i'(R) = \frac{\alpha_i^0 (1+\gamma-\alpha_p^0)}{((1+\gamma-\alpha_p^0) R + \alpha_p^0 r)} \left(\frac{R - R_i^*}{R} \right)$$

and so since $1+\gamma-\alpha_p^0 > 0$ we have $f_i'(R) < 0$ for $R < R_i^*$, $f_i'(R) = 0$ for $R = R_i^*$, and $f_i'(R) > 0$ for $R > R_i^*$. ■

Theorem 29 *For* $v_t^0 \equiv \frac{P_t^c C_{pt}}{M_t^0}, v_t^1 \equiv \frac{P_t^c C_{pt}}{M_t^1}$ *and* $\pi_t^g \equiv \frac{C_{gt}}{C_{pt}}$ *we have*

$$v_t^0 = \frac{R\alpha_p^c}{\alpha_p^0 (1-\tau)}, v_t^1 = \frac{r\alpha_p^c}{\alpha_p^1 (1-\tau)}, \pi_t^g = \frac{\gamma}{\alpha_p^c}.$$

Proof. This follows from Theorems 16, 19, 22, and 24. ■

Table 1: U.S. Sample Means 1990-2001

Cash Velocity v_t^0	Cheque Velocity v_t^1	Nominal Interest Rate R_t	Size of Government π_t^g
$\bar{v}^0 = 9.88$	$\bar{v}^1 = 0.95$	$\bar{R} = 0.057$	$\bar{\pi}^g = 0.32$

Table 2: Implied Parameter Values from Table 1

α_p^0	α_p^1	α_p^c	γ	τ
0.0073	0.0264	0.9663	0.30	0.23

Table 3: U.S. Money Usage According to Income
(from Avery *et al.* (1986) Table 1)

Annual Income (,000)	\$0 – \$10	\$10 – \$20	\$20 – \$30	\$30 – \$50	\$50+
Cash	53%	40%	32%	29%	18%
Cheques and Credit Cards	47%	60%	68%	71%	82%

Table 4: Implied $\phi_i \equiv \alpha_i^1/\alpha_i^0$ from Table 3

Annual Income (\$,000)	\$0 – \$10	\$10 – \$20	\$20 – \$30	\$30 – \$50	\$50+
ϕ_i	0.9	1.5	2.1	2.5	4.6

Table 5: Optimal Monetary Policy R_i^* and Income Tax τ_i^*

Income	ϕ_i	R_i^*	τ_i^*
Cash Only (Friedman Rule)	0.0	0.00000	1.00
\$0 – \$10	0.9	0.00010	0.63
\$10 – \$20	1.5	0.00017	0.54
\$20 – \$30	2.1	0.00023	0.49
\$30 – \$50	2.5	0.00028	0.44
Representative Individual	3.6	0.00041	0.39
\$50+	4.6	0.00052	0.37
Cheque Only (Hyperinflation)	∞	∞	0.23