New Eras, Consumption, and Saving

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Abstract

This paper looks at consumption and saving when the economy is subjected to a non-ergodic regime change. The non-ergodic regime change causes the uncertainty of future income to increase from $O(k)$ to $O\left(k^2\right)$ where k is the forecast horizon. Households have an overwhelming incentive to prevent the $O\left(k^2\right)$ uncertainty from entering the consumption stream, which they do by shifting it into the saving stream. Even though the $O\left(k^2\right)$ uncertainty is transitory, and dies out as $O\left(t^{-\frac{1}{2}}\right)$ where t is the age of the new regime, the effect on wealth accumulation is permanent.

1 Introduction

The most important events in history happen only once: great discoveries such as the electricity or the internet, the beginning and ending of major wars, and the onset of major financial crises such as the Great Depression or the recent 2008 financial crisis. These great events often signal a non-ergodic regime change, sometimes called a New Era. Nobody had seen an electrical era before electricity was understood, and once they were understood the economy would not return to a pre-electrical era. Similarly the 2008 financial crisis was a non-ergodic regime change, due in part to discoveries in the area of modern finance that were not as well understood as once thought.

Non-ergodic regime changes are the economic equivalent of a change in the law of gravity. Because the new regime has never been observed before, the historical record does not reveal the new law-of-motion. To learn the new law-of-motion agents must wait until enough data has been collected from the new regime. In the meantime economic behavior will react to the uncertainty associated with the unknown law-of-motion. This uncertainty acts as a transitory pulse that works its way over time through the economy. The pulse is largest at the beginning of the new regime when agents have the least amount of data. It then steadily decays as more and more data is collected from the new regime. With standard statistical learning this decay will be slow, at a rate of $O\left(t^{-\frac{1}{2}}\right)$ (like standard errors in econometrics) where t is the length of time the new regime has existed. So it may take some time before this pulse of uncertainty has decayed sufficiently to be negligible.

By way of contrast, with ergodic regime changes history continually repeats itself in a way that reveals the underlying law-of-motion for the economy. For example in Hamilton's (1990) Markov switching model of the business cycle, recessions continually follow expansions, which in turn are followed again by recessions so that eventually the historical record reveals the underlying law-of-motion: a Markov switching model.

In this paper we look at the effect of a non-ergodic regime change on consumption, saving, and wealth. We use a version of the model found in Caballero (1991) where the law-of-motion of income Y_t is a random walk with drift μ and where the non-ergodic regime causes an unknown change in μ. We are able to derive a closed-form solution to the model with learning, and from this work out the implications of the pulse of uncertainty caused by the non-ergodic regime change on economic behavior and welfare.

The pulse of uncertainty can be shown to be large in the following general sense: If k is the forecast horizon, then the conditional variance of future income is $O(k)$ when μ is known, but changes to $O(k^2)$ when the non-ergodic regime change occurs. Agents have an overwhelming incentive to prevent this extra $O\left(k^2\right)$ uncertainty from entering their consumption stream. This leads them to shift the entire $O\left(k^2\right)$ uncertainty into the saving stream, so that the uncertainty of future consumption remains as $O(k)$. From the saving stream the $O\left(k^2\right)$ uncertainty enters wealth. Even though the pulse of uncertainty is transitory, dying out as $O\left(t^{-\frac{1}{2}}\right)$, it turns out surprisingly to have a permanent effect on wealth. If we put this into the current post-2008 context, this result would say that even though uncertainty about the nature of post-2008 world will eventually be resolved, this uncertainty will nevertheless have a permanent effect on the economy.

1.1 The Model

At time t the agent's utility is

$$V_t = E_t \sum_{\tau=0}^{T} U(C_{t+\tau}) e^{-r\tau}, \ r > 0 \tag{1}$$

where $U(C)$ is exhibits constant absolute risk aversion ($CARA$) as

$$U(C) = -\exp(-\theta C)/\theta, \ \theta > 0. \tag{2}$$

Income Y_t follows a random walk with drift with an unanticipated non-ergodic regime change at $t = 0$ with

$$Y_t = \mu_t + Y_{t-1} + a_t \text{ where } a_t \sim N\left[0, \sigma^2\right]$$

where

$$\mu_t = \begin{cases} \mu^0 & \text{for } t < 0 \\ \mu^1 & \text{for } t \geq 0 \end{cases}.$$

We assume the old regime has existed for an infinite time span (i.e., all $t < 0$) so that the historical record reveals μ^0 and σ are known. At $t = 0$ a new μ^1 is drawn from the distribution

$$\mu^1 | P_0 \sim N[\hat{\mu}_0^1, \frac{\sigma^2}{t_0}] \tag{3}$$

so that for $t \geq 0$ we have

$$Y_t = \mu^1 + Y_{t-1} + a_t \text{ where } a_t \sim N\left[0, \sigma^2\right].$$

Agents know (3) and so it acts as their prior for μ_1. Here t_0 measures how much agents know about the New Era. If $t_0 = 0$ then they know nothing about μ_1. If $t_0 = \infty$ then $\mu^1 = \hat{\mu}_0^1$ is known.

Let us first examine the nature of uncertainty that agents face in the old regime where μ^0 is known. In this case the only source of uncertainty is unknown future shocks to income. Since

$$Y_{t+k} = Y_t + k\mu^0 + \sum_{j=1}^{k} a_{t+j} = E_t\left[Y_{t+k}\right] + \sum_{j=1}^{k} a_{t+j} \tag{4}$$

so that Y_{t+n} differs from its forecast $E_t\left[Y_{t+k}\right]$ because of the sum of the k unknown future shocks $\sum_{j=1}^{k} a_{t+j}$. Consequently the conditional forecast variance is

$$Var_t[Y_{t+k}] = \sigma^2 k$$

which grows linearly, or $O\left(k\right)$, with the horizon k. The conditional distribution of Y_{t+k} is then

$$Y_{t+k} | Y_t \hat{} \mu^0 \hat{} \sigma \sim N\left[Y_t + k\mu^0, \sigma^2 k\right]. \tag{5}$$

Now let us examine uncertainty after the non-ergodic regime change at $t = 0$. Once this occurs agents begin accumulating sample information $S(t)$ about the New Era as

$$S(t) = [Y_0, Y_1, Y_2, \ldots, Y_t].$$

Since ΔY_t is independent and normally distributed, the posterior for μ^1 comes from standard Bayesian results (see Box and Tiao, 1973 for example) and is

$$\mu^1 \sim N\left[\hat{\mu}_t^1, \frac{\sigma^2}{t + t_o}\right] \tag{6}$$

where the posterior mean $\hat{\mu}_t^1$ is

$$\hat{\mu}_t^1 = \hat{\mu}_{t-1}^1 + \frac{\Delta Y_t - \hat{\mu}_{t-1}^1}{t+t_0} = \bar{\mu}_t^1 + t_o \frac{\hat{\mu}_0^1 - \bar{\mu}_t^1}{t+t_0} \tag{7}$$

where

$$\bar{\mu}_t^1 = \frac{1}{t}\sum_{j=1}^{t} \Delta Y_j = \frac{Y_t - Y_0}{t}$$

is the sample mean of ΔY_t over the New Era.

The pulse of uncertainty $\delta(t)$ discussed in the introduction can be defined as

$$\delta(t) \equiv Var_t\left[\mu^1\right]^{\frac{1}{2}} = \sqrt{\frac{\sigma^2}{t+t_o}}.$$

This pulse is at a maximum at $t = 0$, the beginning of the New Era, with $\delta(0) = \frac{\sigma}{\sqrt{t_o}}$, and declines slowly but monotonically thereafter at the parametric rate as $\delta(t) = O\left(t^{-\frac{1}{2}}\right)$. The slow parametric rate $O\left(t^{-\frac{1}{2}}\right)$ is characteristic of statistical learning.

The fact that μ^1 is unknown fundamentally alters the nature of the uncertainty of future income. To see this note that from (4), (6) and the fact that μ^1 is independent of future shocks that

$$Var_t[Y_{t+k}] = Var_t\left[\sum_{j=1}^{k} a_{t+j}\right] + k^2 Var_t[\mu^1] = \sigma^2 k + \frac{\sigma^2}{t+t_0}k^2. \tag{8}$$

The posterior for Y_{t+k} conditional on the information at t is

$$Y_{t+k}|P_0\hat{\ }S(t)\hat{\ }\sigma \sim N\left[Y_t + k\hat{\mu}_t^1, \sigma^2 k + \frac{\sigma^2}{t+t_0}k^2\right]. \tag{9}$$

Comparing (9) with (5) we see that the conditional variance goes from being $O(k)$ when μ^0 is known in the old regime to $O(k^2)$ in the New Era where μ^1 is unknown. For $k > t + t_0$ parameter uncertainty will dominate unknown future shocks as a source of uncertainty about future income.

The result of $O\left(k^2\right)$ uncertainty when μ^1 is unknown can be shown to hold under very general assumptions, as shown by Sampson (1991). This is because of the simple fact that with a trend $\mu^1 k$ we have

$$Var_t[\mu^1 k] = Var_t[\mu^1]k^2.$$

As long as $Var_t[\mu^1] > 0$ this $O\left(k^2\right)$ uncertainty is the dominant source of uncertainty.

1.2 The Welfare Cost of Uncertainty when Consumption Equals Income

Here we will show that agents will have an overwhelming incentive to prevent the $O\left(k^2\right)$ uncertainty in the income stream from entering the consumption stream C_t. To do this suppose that $C_t = Y_t$. (Similar results will also hold if $C_t = \alpha Y_t$ instead.)

Combining (1) and (9) it follows that welfare at time t is

$$V_{1t}(t_0, T) = U(C_t) \sum_{\tau=0}^{T} \exp\left(-\lambda_{1t}\tau + \lambda_{2t}\tau^2\right) \tag{10}$$

where

$$\lambda_{1t} = r + \theta(\hat{\mu}_t^1 - \theta\frac{\sigma^2}{2}) \text{ and } \lambda_{2t} = \frac{\theta^2\sigma^2}{2\left(t+t_0\right)} > 0. \tag{11}$$

Since $\lambda_{2t}\tau^2 > 0$ and increases as $O\left(\tau^2\right)$, it must eventually dominate $-\lambda_{1t}\tau$ for large enough T. Thus

$$\lim_{T\to\infty} V_{1t}(t_0, T) = -\infty \tag{12}$$

so that for a long enough time T the uncertainty regarding μ^1 dominates all other factors influencing welfare.

The welfare cost of uncertainty can be measured by the amount of consumption $\tilde{\gamma}_t(t_o, T)$ an agent would be willing to sacrifice in return for knowing that future income will be $Y_t + k\hat{\mu}_t^1$ with certainty. This is defined by

$$U(C_t) \sum_{\tau=0}^{T} \exp\left(-\lambda_{1t}\tau + \lambda_{2t}\tau^2\right) = U\left(C_t - \tilde{\gamma}_t(t_o, T)\right) \sum_{\tau=0}^{T} \exp\left(-\lambda_{1t}\tau\right)$$

so that

$$\tilde{\gamma}_t(t_o, T) = \frac{1}{\theta} \ln\left[\frac{\sum_{\tau=0}^{T} \exp\left(-\lambda_{1t}\tau + \lambda_{2t}\tau^2\right)}{\sum_{\tau=0}^{T} \exp\left(-\lambda_{1t}\tau\right)}\right]. \tag{13}$$

We then see that

$$\lim_{T\to\infty} \tilde{\gamma}_t(t_o, T) = \infty \tag{14}$$

so that the welfare costs of an unknown μ^1 increase without bound.

2 Introducing Saving

Suppose that the agent can borrow and save at a constant real rate of interest r so that wealth A_t evolves as

$$A_t = e^r(A_{t-1} + Y_{t-1} - C_{t-1}),\ r > 0,\ A_0 = 0. \tag{15}$$

There is no bequest motive so that $A_{T+1} = 0$ or alternatively

$$C_T = Y_T + A_T. \tag{16}$$

Maximizing welfare at time t leads to the standard Euler equation

$$U'(C_t) = E_t\left[U'(C_{t+1})\right]$$

or

$$\exp(-\theta C_t) = E_t\left[\exp(-\theta C_{t+1})\right]. \tag{17}$$

In the appendix it is shown that

$$C_t = Y_t + \alpha_t(T)A_t + \beta_t(T)\hat{\mu}_t^1 - \gamma_t(t_o, T) \tag{18}$$

where

$$\begin{aligned}
\alpha_t(T) &= \frac{1-e^{-r}}{1-e^{-r(T+1-t)}} > 0 \qquad (19)\\
\beta_t(T) &= \frac{e^{-r}}{1-e^{-r}}\left[\frac{1-e^{-r(T-t)}-(T-t)\,e^{-r(T-t)}\,(1-e^{-r})}{1-e^{-r(T+1-t)}}\right] > 0\\
\gamma_t(t_o,T) &= \frac{\alpha_t(T)\theta\sigma^2}{2}\sum_{k=1}^{T-t}\frac{e^{-rk}}{\alpha_{t+k+1}(T)}\left(1+\frac{\beta_{t+k+1}(T)}{t+t_o+k}\right)^2\left(1+\frac{1}{t+t_{o+k-1}}\right) > 0.
\end{aligned}$$

From (18) it follows that saving: $S_t \equiv Y_t - C_t$ is

$$S_t = -\alpha_t(T)A_t - \beta_t(T)\hat{\mu}_t^1 + \gamma_t(t_o, T). \tag{20}$$

The value function can by determined by using the law of iterated expectations and the fact that the marginal utility of consumption is proportional to the utility of consumption to obtain

$$V_{2t}\left(t_o, T\right) = \frac{U\left(C_t\right)}{\alpha_t(T)}. \tag{21}$$

An alternative interpretation of $\gamma_t(t_o, T)$ is the welfare cost of uncertain future consumption. In particular at time t the individual would be indifferent between his present random income process and a deterministic income process of

$$Y_{t+\tau} = Y_t + \tau\hat{\mu}_t^1 - \gamma_t(t_o, T),\ \tau = 0, 1, 2, \ldots, T-t. \tag{22}$$

Thus $\gamma_t(t_o, T)$ can be compared with $\tilde{\gamma}_t(t_o, T)$ in (13) to assess the relative welfare costs of future income uncertainty with and without precautionary saving.

From the Euler equation in (17) and the conditional normality of C_{t+k} it follows that

$$E_t[\exp(-\theta C_{t+k})] = \exp(-\theta C_t)$$

so that from the conditional normality of C_{t+k} it follows that

$$Var_t[C_{t+k}] = \frac{2}{\theta}E_t\left[C_{t+k} - C_t\right]. \tag{23}$$

Now consider what happens if we allow an infinite horizon $T = \infty$. (Recall that $T = \infty$ was not possible when $C_t = Y_t$ without incurring an infinite welfare loss.) From (18) with $T = \infty$ we have

$$\begin{aligned} E_t[C_{t+k} - C_t] &= k\hat{\mu}_t^1 + (1 - e^{-r})E_t[A_{t+k} - A_t] + \gamma_t(t_0) - \gamma_{t+k}(t_0) \\ &= (e^r - 1)\sum_{j=0}^{k-1} \gamma_{t+j}(t_0) + \gamma_t(t_0) - \gamma_{t+k}(t_0) \end{aligned}$$

where the last equality follows from using (29) to evaluate $E_t[A_{t+k} - A_t]$. Hence $Var_t[C_{t+k}]$ is

$$Var_t[C_{t+k}] = \frac{2}{\theta}\left((e^r - 1)\sum_{j=0}^{k-1} \gamma_{t+j}(t_0) + \gamma_t(t_0) - \gamma_{t+k}(t_0)\right). \qquad (24)$$

Exponential discounting insures that as $T \to \infty$, that the limits of $\alpha_t(T), \beta_t(T), \gamma_t(t_o, T)$ are all finite, as given respectively by

$$\alpha = 1 - e^{-r}, \beta = \frac{e^{-r}}{1 - e^{-r}}, \gamma_t(t_o) = \frac{\theta\sigma^2}{2}\sum_{k=1}^{T-t} e^{-rk}\left(1 + \frac{\beta}{t + t_o + k}\right)^2\left(1 + \frac{1}{t + t_{o+k-1}}\right).$$

Since $\gamma_t(t_o) < \infty$ it follows that the welfare costs of uncertain future income are finite, even with an infinite horizon.

The infinite horizon $T = \infty$ can only be possible if the agent has shifted the $O(k^2)$ uncertainty in the income stream Y_t entirely to the saving stream S_t. We now show this. From the Euler equation in (17) it follows that

$$E_t[\exp(-\theta C_{t+k})] = \exp(-\theta C_t)$$

and hence from the conditional normality of C_{t+k}

$$Var_t[C_{t+k}] = \frac{2}{\theta}E_t\left[C_{t+k} - C_t\right]$$

where $E_t\left[C_{t+k} - C_t\right]$ which is $O\left(k\right)$. In particular it can be shown that

$$Var_t[C_{t+k}] \leq \frac{2}{\theta}(e^r - 1)\gamma_t(t_o)\left(k + 1\right)$$

so $Var_t[C_{t+k}]$ is $O\left(k\right)$ even though $Var_t\left[Y_{t+k}\right]$ is $O\left(k^2\right)$. Since $Y_t = C_t + S_t$, it follows that $Var_t[S_{t+k}] = O(k^2)$. From (20) and the fact that $\hat{\mu}_{t+k}^1$ is $O_p(1)$, it follows that $Var_t[A_{t+k}]$ is $O(k^2)$, so that the wealth stream also inherits the $O\left(k^2\right)$ uncertainty.

While the particular details will change, it seems that a similar results will hold for more realistic models of consumption and saving that do not have a closed-form solution. For example consider the case of $CRRA$ preferences. The argument for an infinite welfare loss from an unknown μ^1 then follows using exactly the same argument in the previous section using a log-linear version of the model. When saving is possible there will be no closed-form solution as we have, but from the Euler equation

$$Var_t[\ln(C_{t+k})] = \frac{2}{\theta} E_t[\ln(C_{t+k}) - \ln(C_t)]$$

where θ is now interpreted as the coefficient of relative risk aversion. Thus as long as the forecast of future log consumption increases linearly, which is reasonable when there is a trend, then $E_t[\ln(C_{t+k})]$ will be $O(k)$ and hence $Var_t[\ln(C_{t+k})]$ will be $O(k)$.

3 Wealth Accumulation

In this section we show that at least some of the wealth accumulation due to the unknown μ^1 is permanent; that is the level of wealth will not move to a level that would have occurred if there had been no non-ergodic regime change. From (18) with $T = \infty$ consumption for someone of generation t will be

$$C_t = Y_t + \alpha A_t + \beta \hat{\mu}_t^1 - \gamma_t(t_o) \tag{25}$$

or, using (22) and $S_t \equiv Y_t - C_t$

$$A_t + S_t = e^{-r} A_t - \beta \hat{\mu}_t^1 + \gamma_t(t_o). \tag{26}$$

Since $A_{t+1} = e^r(A_t + S_t)$

$$A_{t+1} - A_t = -\frac{\hat{\mu}_t^1}{1 - e^{-r}} + e^r \gamma_t(t_o) \tag{27}$$

and since $A_0 = 0$

$$A_t = -\frac{\mu t}{1 - e^{-r}} - \sum_{k=1}^{t} \left(\frac{\mu - \hat{\mu}_{t-k}^1}{1 - e^{-r}} + e^r \gamma_{t-k}(t_o) \right). \tag{28}$$

The term $-\frac{\mu t}{1-e^{-r}}$ reflects the amount of wealth that would be accumulated without any uncertainty. The two terms in the brackets of the summation in (28) capture the effect of uncertainty at time $t-k$ on wealth at time t. The first of these $\frac{\mu - \hat{\mu}_{t-k}^1}{1-e^{-r}}$ reflects the past error made in the estimation of μ^1 at $t-k$ while the second: $e^r \gamma_{t-k}(t_o)$ reflects the precautionary saving motive at time $t-k$.

From either (27) or (28) it follows that some of the wealth accumulation is permanent; that is, the effects of $\frac{\mu - \hat{\mu}_{t-k}^1}{1-e^{-r}}$ and $e^r \gamma_{t-k}(t_o)$ on future wealth do not diminish over time. Thus even though the agent is learning and hence his uncertainty is diminishing, there is no tendency for him to attempt to undo the effects of past errors or saving decisions on his present level of wealth.

It is possible to decompose $\gamma_t(t_o)$ accordingly as

$$\gamma_t(t_o) = \gamma_1 + \gamma_{2t}(t_o) \tag{29}$$

where

$$\gamma_1 = \frac{\beta\theta\sigma^2}{2} > 0 \tag{30}$$

reflects unknown future shocks and

$$\gamma_{2t}(t_o) = \frac{\theta\sigma^2}{2}\sum_{k=1}^{\infty} e^{-rk}\left(\left(1+\frac{\beta}{t+t_o+k}\right)^2\left(1+\frac{1}{t+t_o+k-1}\right)-1\right)$$

reflects an unknown μ^1.

It also is possible to decompose wealth along similar lines. From (28), (29), and (30)

$$A_t = -\frac{\mu^1 - \frac{\theta\sigma^2}{2}}{1-e^{-r}}t - \sum_{k=1}^{t}\frac{\mu^1 - \hat{\mu}^1_{t-k}}{1-e^{-r}} + e^r\sum_{k=1}^{t}\gamma_{2t-k}(t_o) \tag{31}$$

so that the effect of unknown future shocks on wealth is to adjust the effective μ^1 down by $\frac{\theta\sigma^2}{2}$, the effect of past estimation errors is captured by the second term and the effect of the precautionary motive due to an unknown μ^1 is captured by the third term.

4 Conclusions

Households facing a non-ergodic regime change in the income process will have an overwhelming incentive to shift the resulting $O(k^2)$ uncertainty into the saving stream. When this is done it turns out to have a permanent effect on wealth. In the model we have assumed a constant rate of interest, and so this is only one mechanism, and not the entire story of how an economy will react to a non-ergodic regime change. In a general equilibrium setting somebody will have to be willing to take on the $O(k^2)$ uncertainty, so that interest rates would react, and this will have repercussions throughout the economy. What is clear is that the pulse of uncertainty from a non-ergodic regime change has the potential to have big economic consequences.

While the model used in this paper is specific and special, the methodology is general. Wherever there is a non-ergodic regime change there will be a spike in uncertainty that then decays over time. Take one's favourite model, subject it to this spike, and trace out the implications for economic behavior.

5 References

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6 Appendix

From the Euler equation (17) for $t-1$ substitute (18) in for C_t to yield

$$\begin{aligned} \exp(-\theta C_t) &= E_{t-1}\left[\exp\left(-\theta(Y_t+\alpha_t A_t+\beta_t\hat{\mu}_t^1-\gamma_t)\right)\right] \quad (32)\\ &= E_{t-1}\left[\exp\left(\begin{array}{c} -\theta(Y_{t-1}+\Delta Y_t+\alpha_t e^r\left(A_{t-1}+Y_{t-1}-C_{t-1}\right) \\ +\beta_t\left(\hat{\mu}_{t-1}^1+\frac{\Delta Y_t-\hat{\mu}_{t-1}^1}{t+t_0}\right)-\gamma_t) \end{array}\right)\right] \end{aligned}$$

where the second equality follows from $Y_t = Y_{t-1}+\Delta Y_t$, (7) and (15) and where we suppress the dependence of the coefficients on t_0 and T. Collecting the terms which are in the information set and those which are not, we find that the term not in the information set will be

$$E_{t-1}\left[\exp\left(-\theta\left(1+\frac{\beta_t}{t+t_0}\right)\right)\Delta Y_t\right] = \exp\left[\begin{array}{c} \left(-\theta\left(1+\frac{\beta_t}{t+t_0}\right)\right)\hat{\mu}_{t-1}^1 \\ +\frac{\theta^2\sigma^2}{2}\left(\left(1+\frac{\beta_t}{t+t_0}\right)^2\left(1+\frac{1}{t+t_0-1}\right)\right) \end{array}\right] \quad (33)$$

since the distribution of ΔY_t conditional on the information set at $t-1$ is

$$\Delta Y_t | S\left(t-1\right) \hat{\ } P_0 \hat{\ } \sigma \sim N\left[\hat{\mu}_{t-1}^1, \sigma^2\left(1+\frac{1}{t+t_0-1}\right)\right]$$

which follows from (9) with t replaced by $t-1$ and $k=1$.

Using (33) in (32) and solving for C_{t-1} then yields

$$C_{t-1} = Y_{t-1}+\alpha_{t-1}A_{t-1}+\beta_{t-1}\hat{\mu}_{t-1}^1-\gamma_{t-1} \quad (34)$$

where

$$\alpha_{t-1} = \frac{\alpha_t e^r}{1+\alpha_t e^r}, \beta_{t-1} = \frac{1+\beta_t}{1+\alpha_t e^r}, \gamma_{t-1} = \frac{\gamma_t+\frac{\theta^2\sigma^2}{2}\left(\left(1+\frac{\beta_t}{t+t_0}\right)^2\left(1+\frac{1}{t+t_0-1}\right)\right)}{1+\alpha_t e^r} \quad (35)$$

From (35) it follows that

$$\alpha_{t-1}^{-1} = e^{-r}\alpha_t^{-1}+1 \quad (36)$$

so that using $\alpha_T = 1$ and solving (36) forwards results in the first equation in (19). To the rest note that from (35)

$$\frac{1}{1+\alpha_t e^r} = \frac{e^{-r}\alpha_{t-1}}{\alpha_t} \quad (37)$$

so that if $\tilde{\beta}_t = \frac{\beta_t}{\alpha_t}$ and $\tilde{\gamma}_t = \frac{\gamma_t}{\alpha_t}$ then β_t and γ_t in (35) can be rewritten as

$$\tilde{\beta}_{t-1} = e^{-r}\tilde{\beta}_t + \frac{e^{-r}}{\alpha_t}, \tilde{\gamma}_{t-1} = e^{-r}\tilde{\gamma}_t + \frac{e^{-r}}{\alpha_t}\frac{\theta^2\sigma^2}{2}\left(\left(1+\frac{\beta_t}{t+t_0}\right)^2\left(1+\frac{1}{t+t_0-1}\right)\right). \quad (38)$$

Again solving forwards and using $\tilde{\beta}_T = \tilde{\gamma}_T = 0$ then yields, after some straightforward manipulation, the required results.